Math 21A
Kouba
The Mean Value Theorem (MVT) and Other Important Theorems

Definition : Function f takes on its *maximum* value at $x = c$ if $f(x) < f(c)$ for $x \neq c$. Function f takes on its *minimum* value at $x = c$ if $f(c) < f(x)$ for $x \neq c$.

Theorem A : If function f is differentiable at $x = c$, then f is continuous at $x = c$.

Theorem B : Assume that function f is differentiable and takes on its maximum value at $x = c$. Then $f'(c) = 0$.

Theorem C : Assume that function f is differentiable and takes on its minimum value at $x = c$. Then $f'(c) = 0$.

Rolle's Theorem : Assume that function f is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$. Then there is at least one number c, $a < c < b$, so that $f'(c) = 0$.

PROOF : Since f is continuous on a closed interval $[a, b]$ f has a maximum value M and a minimum value m. This follows from the Maximum/Minimum Value Theorems dicussed earlier in this course.

case 1. If $m = M$, then $f(x) = k$ for some constant k and all values x in $[a, b]$. Thus, $f'(x) = 0$ for all values of x in $[a, b]$. It follows that $f'(c) = 0$ for some value of c, $a < c < b$.

case 2. If $m < M$, then both m and M cannot occur at endpoints a and b since $f(a) = f(b)$. Thus, at least one occurs in the interior of the interval at $x = c$. It follows from Theorems B and C that $f'(c) = 0$. QED

Mean Value Theorem (MVT) : Assume that function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one number c, $a < c < b$, so that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

PROOF : The equation of line L in the diagram is

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} ,$$

so that

$$y = \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a) .$$

Define a new function

$$s(x) = f(x) - y = f(x) - \left\{ \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a) \right\} .$$

This function is differentiable on the open interval (a, b) since it is the difference of differentiable functions. This function is continuous on the closed interval $[a, b]$ since it is the difference of continuous functions. In addition, $s(a) = 0$ and $s(b) = 0$. It follows from Rolle's Theorem that there exists a number c, $a < c < b$, so that $s'(c) = 0$. Since

$$s'(x) = f'(x) - \left\{ \frac{f(b) - f(a)}{b - a} \cdot (1) + (0) \right\} = 0 ,$$

it follows that

$$s'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \longrightarrow$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} . \qquad \text{QED}$$

Theorem D : Assume that $f'(x) = 0$ for all values of x in the closed interval $[a, b]$. Then $f(x) = k$, a constant function on $[a, b]$.

PROOF : Consider any two arbitrary x-values w and z in $[a, b]$ with $w < z$. Consider the restriction of f to the new interval $[w, z]$. Since f is differentiable on the closed interval $[w, z]$ (and hence on the open interval (w, z), it follows from Theorem A that f is continuous on the open interval (w, z). By the MVT there is at least one number c, $w < c < z$, so that

$$f'(c) = \frac{f(z) - f(w)}{z - w} \quad \longrightarrow$$

$$\frac{f(z) - f(w)}{z - w} = 0 \text{ (Since f'(c)=0)} \quad \longrightarrow$$

$$f(z) - f(w) = 0 \quad \longrightarrow$$

$$f(z) = f(w) \ .$$

Since w and z were chosen arbitrarily, it must be that $f(x) = k$ for some constant k and for all values of x in the closed interval $[a, b]$. QED

Theorem E : Assume that $f'(x) = g'(x)$ for all values of x in the closed interval $[a, b]$. Then $f(x) = g(x) + c$ for some constant c.

PROOF : Since $f'(x) = g'(x) \quad \longrightarrow$

$$f'(x) - g'(x) = 0 \quad \longrightarrow$$

$$D(f(x) - g(x)) = 0 \quad \longrightarrow$$

$$f(x) - g(x) = c \text{ for some constant } c \text{ (by Theorem D)} \quad \longrightarrow$$

$$f(x) = g(x) + c \text{ for some constant } c \ . \qquad \text{QED}$$