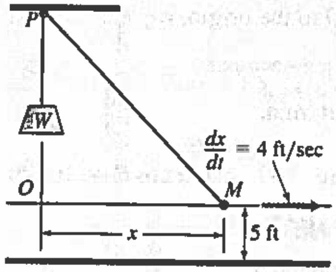
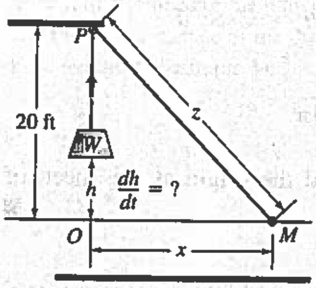


(a)



(b)

FIGURE 3.49 A worker at M walks to the right, pulling the weight W upward as the rope moves through the pulley P (Example 6).

EXAMPLE 6 Figure 3.49a shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line PW at the rate of 4 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from PW?

Solution We let OM be the horizontal line of length x ft from a point O directly below the pulley to the worker's hand M at any instant of time (Figure 3.49). Let h be the height of the weight W above O, and let z denote the length of rope from the pulley P to the worker's hand. We want to know dh/dt when $x = 21$ given that $dx/dt = 4$. Note that the height of P above O is 20 ft because O is 5 ft above the ground. We assume the angle at O is a right angle.

At any instant of time t we have the following relationships (see Figure 3.49b):

$$20 - h + z = 45 \qquad \text{Total length of rope is 45 ft.}$$

$$20^2 + x^2 = z^2. \qquad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2. \tag{1}$$

Differentiating both sides with respect to t gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt},$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \frac{x}{25 + h}\frac{dx}{dt}. \tag{2}$$

Since we know dx/dt, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

so that

$$(25 + h)^2 = 841, \qquad \text{or} \qquad 25 + h = 29.$$

Equation (2) now gives

$$\frac{dh}{dt} = \frac{21}{29}\cdot 4 = \frac{84}{29} \approx 2.9 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft. ■

Exercises 3.10

1. **Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.
2. **Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates dS/dt to dr/dt.
3. Assume that $y = 5x$ and $dx/dt = 2$. Find dy/dt.
4. Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find dx/dt.
5. If $y = x^2$ and $dx/dt = 3$, then what is dy/dt when $x = -1$?
6. If $x = y^3 - y$ and $dy/dt = 5$, then what is dx/dt when $y = 2$?
7. If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is dy/dt when $x = 3$ and $y = -4$?
8. If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?
9. If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find dL/dt when $x = 5$ and $y = 12$.
10. If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.

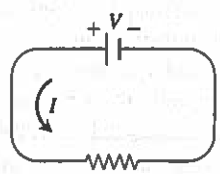
11. If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's
   a. surface area change?
   b. volume change?

12. A cube's surface area increases at the rate of 72 in^2/sec. At what rate is the cube's volume changing when the edge length is $x = 3$ in?

13. **Volume** The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   a. How is dV/dt related to dh/dt if r is constant?
   b. How is dV/dt related to dr/dt if h is constant?
   c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

14. **Volume** The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
   a. How is dV/dt related to dh/dt if r is constant?
   b. How is dV/dt related to dr/dt if h is constant?
   c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

15. **Changing voltage** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in seconds.



   a. What is the value of dV/dt?
   b. What is the value of dI/dt?
   c. What equation relates dR/dt to dV/dt and dI/dt?
   d. Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps. Is R increasing, or decreasing?

16. **Electrical power** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.
   a. How are dP/dt, dR/dt, and dI/dt related if none of P, R, and I are constant?
   b. How is dR/dt related to dI/dt if P is constant?

17. **Distance** Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.
   a. How is ds/dt related to dx/dt if y is constant?
   b. How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
   c. How is dx/dt related to dy/dt if s is constant?

18. **Diagonals** If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.
   a. Assuming that x, y, and z are differentiable functions of t, how is ds/dt related to dx/dt, dy/dt, and dz/dt?
   b. How is ds/dt related to dy/dt and dz/dt if x is constant?
   c. How are dx/dt, dy/dt, and dz/dt related if s is constant?

19. **Area** The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is

$$A = \frac{1}{2}ab\sin\theta.$$

   a. How is dA/dt related to $d\theta/dt$ if a and b are constant?
   b. How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
   c. How is dA/dt related to $d\theta/dt$, da/dt, and db/dt if none of a, b, and θ are constant?

20. **Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

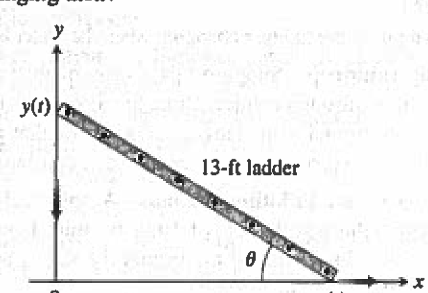
21. **Changing dimensions in a rectangle** The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

22. **Changing dimensions in a rectangular box** Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

23. **A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.
   a. How fast is the top of the ladder sliding down the wall then?
   b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
   c. At what rate is the angle θ between the ladder and the ground changing then?



24. **Commercial air traffic** Two commercial airplanes are flying at an altitude of 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5

nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

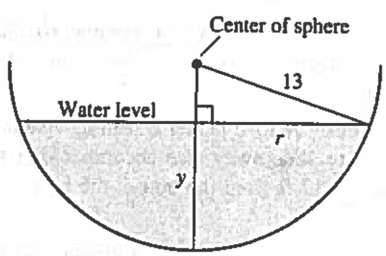
25. **Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

26. **Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

27. **A growing sand pile** Sand falls from a conveyor belt at the rate of $10\ m^3/min$ onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in centimeters per minute.

28. **A draining conical reservoir** Water is flowing at the rate of $50\ m^3/min$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
    a. How fast (centimeters per minute) is the water level falling when the water is 5 m deep?
    b. How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

29. **A draining hemispherical reservoir** Water is flowing at the rate of $6\ m^3/min$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.

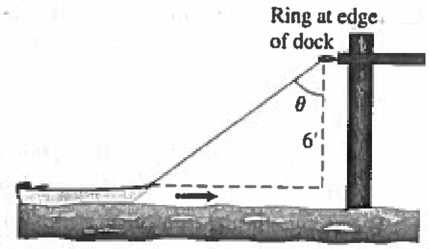


    a. At what rate is the water level changing when the water is 8 m deep?
    b. What is the radius r of the water's surface when the water is y m deep?
    c. At what rate is the radius r changing when the water is 8 m deep?

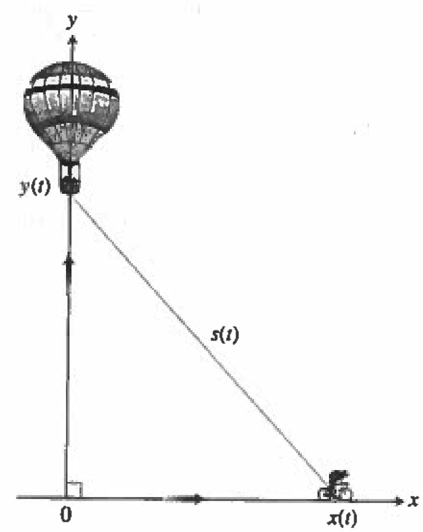
30. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

31. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi\ ft^3/min$. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

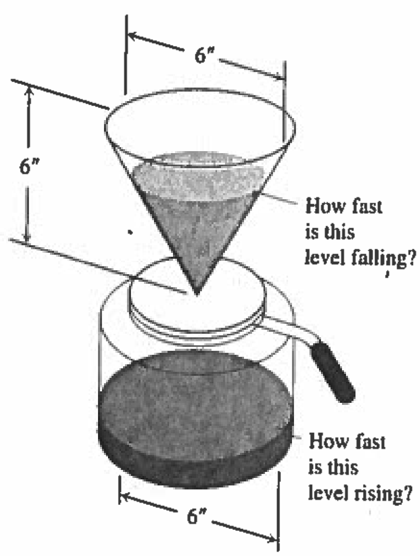
32. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.
    a. How fast is the boat approaching the dock when 10 ft of rope are out?
    b. At what rate is the angle θ changing at this instant (see the figure)?



33. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?



34. **Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10\ in^3/min$.
    a. How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?
    b. How fast is the level in the cone falling then?



35. **Cardiac output** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

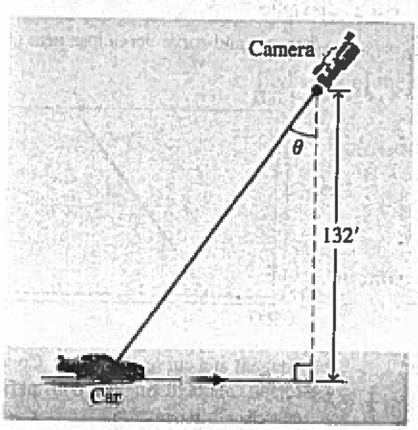
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

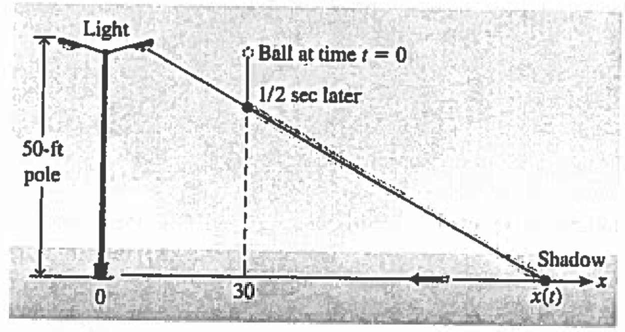
36. **Moving along a parabola** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

37. **Motion in the plane** The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

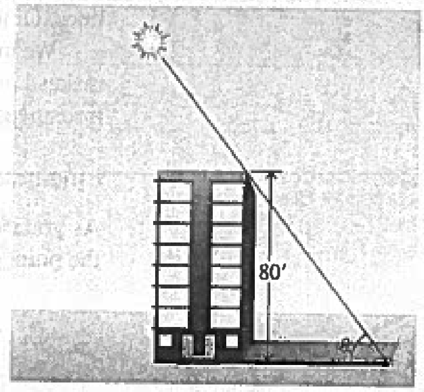
38. **Videotaping a moving car** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec), as shown in the accompanying figure. How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



39. **A moving shadow** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



40. **A building's shadow** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27°$/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



41. **A melting ice layer** A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

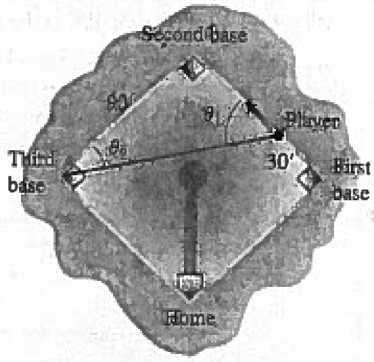
42. **Highway patrol** A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

43. **Baseball players** A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

a. At what rate is the player's distance from third base changing when the player is 30 ft from first base?

b. At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?

c. The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?



44. **Ships** Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

45. **Clock's moving hands** At what rate is the angle between a clock's minute and hour hands changing at 4 o'clock in the afternoon?

46. **Oil spill** An explosion at an oil rig located in gulf waters causes an elliptical oil slick to spread on the surface from the rig. The slick is a constant 9 in. thick. After several days, when the major axis of the slick is 2 mi long and the minor axis is 3/4 mi wide, it is determined that its length is increasing at the rate of 30 ft/hr, and its width is increasing at the rate of 10 ft/hr. At what rate (in cubic feet per hour) is oil flowing from the site of the rig at that time?

3.11 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 10.

We introduce new variables dx and dy, called *differentials*, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement, which then provides for a precise proof of the Chain Rule (Section 3.6).

Linearization

As you can see in Figure 3.50, the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line

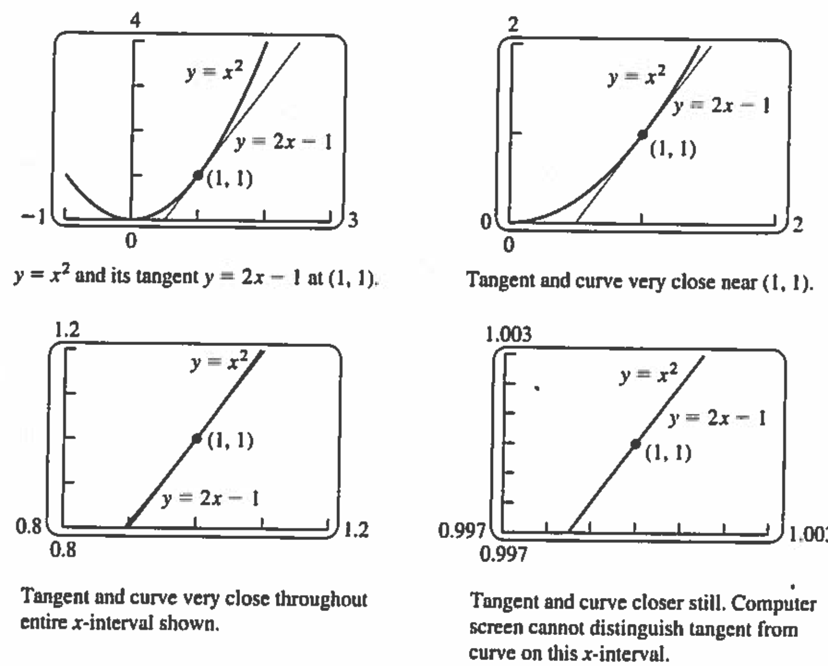


FIGURE 3.50 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right). \tag{3}$$

Equation (3) expresses the increase in mass that results from the added velocity v. ■

Converting Mass to Energy

Equation (3) derived in Example 9 has an important interpretation. In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the object, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0 (0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/sec, we see that a small change in mass can create a large change in energy.

Exercises 3.11

Finding Linearizations

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$
4. $f(x) = \sqrt[3]{x}, \quad a = -8$
5. $f(x) = \tan x, \quad a = \pi$
6. **Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

   a. $\sin x$ b. $\cos x$ c. $\tan x$ d. e^x e. $\ln(1 + x)$

Linearization for Approximation

In Exercises 7–14, find a linearization at a suitably chosen integer near a at which the given function and its derivative are easy to evaluate.

7. $f(x) = x^2 + 2x, \quad a = 0.1$
8. $f(x) = x^{-1}, \quad a = 0.9$
9. $f(x) = 2x^2 + 3x - 3, \quad a = -0.9$
10. $f(x) = 1 + x, \quad a = 8.1$
11. $f(x) = \sqrt[3]{x}, \quad a = 8.5$
12. $f(x) = \dfrac{x}{x + 1}, \quad a = 1.3$
13. $f(x) = e^{-x}, \quad a = -0.1$
14. $f(x) = \sin^{-1} x, \quad a = \pi/12$
15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.
16. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

    a. $f(x) = (1 - x)^6$ b. $f(x) = \dfrac{2}{1 - x}$

    c. $f(x) = \dfrac{1}{\sqrt{1 + x}}$ d. $f(x) = \sqrt{2 + x^2}$

    e. $f(x) = (4 + 3x)^{1/3}$ f. $f(x) = \sqrt[3]{\left(1 - \dfrac{x}{2 + x}\right)^2}$
17. **Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

    a. $(1.0002)^{50}$ b. $\sqrt[3]{1.009}$
18. Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x + 1}$ and $\sin x$ at $x = 0$?

Derivatives in Differential Form

In Exercises 19–38, find dy.

19. $y = x^3 - 3\sqrt{x}$

20. $y = x\sqrt{1 - x^2}$

21. $y = \dfrac{2x}{1 + x^2}$

22. $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

23. $2y^{3/2} + xy - x = 0$

24. $xy^2 - 4x^{3/2} - y = 0$

25. $y = \sin(5\sqrt{x})$

26. $y = \cos(x^2)$

27. $y = 4\tan(x^3/3)$

28. $y = \sec(x^2 - 1)$

29. $y = 3\csc\left(1 - 2\sqrt{x}\right)$

30. $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

31. $y = e^{\sqrt{x}}$

32. $y = xe^{-x}$

33. $y = \ln(1 + x^2)$

34. $y = \ln\left(\dfrac{x + 1}{\sqrt{x - 1}}\right)$

35. $y = \tan^{-1}(e^{x^2})$

36. $y = \cot^{-1}\left(\dfrac{1}{x^2}\right) + \cos^{-1} 2x$

37. $y = \sec^{-1}(e^{-x})$

38. $y = e^{\tan^{-1}\sqrt{x^2+1}}$

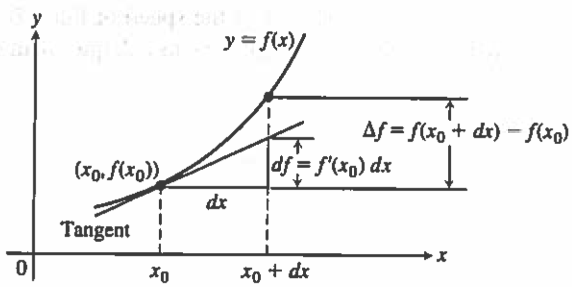
Approximation Error

In Exercises 39–44, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a. the change $\Delta f = f(x_0 + dx) - f(x_0)$;

b. the value of the estimate $df = f'(x_0)\,dx$; and

c. the approximation error $|\Delta f - df|$.



39. $f(x) = x^2 + 2x, \quad x_0 = 1, \quad dx = 0.1$

40. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

41. $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$

42. $f(x) = x^4, \quad x_0 = 1, \quad dx = 0.1$

43. $f(x) = x^{-1}, \quad x_0 = 0.5, \quad dx = 0.1$

44. $f(x) = x^3 - 2x + 3, \quad x_0 = 2, \quad dx = 0.1$

Differential Estimates of Change

In Exercises 45–50, write a differential formula that estimates the given change in volume or surface area.

45. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$

46. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

47. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

48. The change in the lateral surface area $S = \pi r\sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change

49. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change

50. The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change

Applications

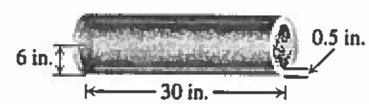
51. The radius of a circle is increased from 2.00 to 2.02 m.

a. Estimate the resulting change in area.

b. Express the estimate as a percentage of the circle's original area.

52. The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-sectional area?

53. Estimating volume Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in.



54. Estimating height of a building A surveyor, standing 30 ft from the base of a building, measures the angle of elevation to the top of the building to be 75°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?

55. The radius r of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's

a. circumference?

b. area?

56. The edge x of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's

a. surface area?

b. volume?

57. Tolerance The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

58. Tolerance

a. About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

b. About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

59. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

60. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.

61. The effect of flight maneuvers on the heart The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the weight density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \text{ } (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2 \text{ ft/sec}^2$, with the effect the same change dg would have on Earth, where $g = 32 \text{ ft/sec}^2$. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth}.

62. **Drug concentration** The concentration C in milligrams per milliliter (mg/ml) of a certain drug in a person's bloodstream t hrs after a pill is swallowed is modeled by

$$C(t) = 1 + \frac{4t}{1 + t^3} - e^{-0.06t}.$$

Estimate the change in concentration when t changes from 20 to 30 min.

63. **Unclogging arteries** The formula $V = kr^4$, discovered by the physiologist Jean Poiseuille (1797–1869), allows us to predict how much the radius of a partially clogged artery has to be expanded in order to restore normal blood flow. The formula says that the volume V of blood flowing through the artery in a unit of time at a fixed pressure is a constant k times the radius of the artery to the fourth power. How will a 10% increase in r affect V?

64. **Measuring acceleration of gravity** When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

a. With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).

b. If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.

c. A clock with a 100-cm pendulum is moved from a location where $g = 980 \text{ cm/sec}^2$ to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

65. **Quadratic approximations**

a. Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i) $Q(a) = f(a)$

ii) $Q'(a) = f'(a)$

iii) $Q''(a) = f''(a)$.

Determine the coefficients b_0, b_1, and b_2.

b. Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

T c. Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point $(0, 1)$. Comment on what you see.

T d. Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

T e. Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

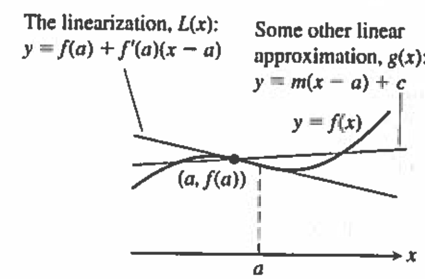
f. What are the linearizations of f, g, and h at the respective points in parts (b), (d), and (e)?

66. **The linearization is the best linear approximation** Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

1. $E(a) = 0$ — The approximation error is zero at $x = a$.

2. $\lim_{x \to a} \frac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



67. **The linearization of 2^x**

a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.

T b. Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

68. **The linearization of $\log_3 x$**

a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.

T b. Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

COMPUTER EXPLORATIONS

In Exercises 69–74, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval I. Perform the following steps:

a. Plot the function f over I.

b. Find the linearization L of the function at the point a.

c. Plot f and L together on a single graph.

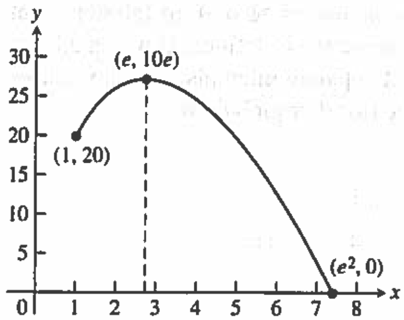


FIGURE 4.8 The extreme values of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$ occur at $x = e$ and $x = e^2$ (Example 3).

Solution Figure 4.8 suggests that f has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation.

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative is

$$f'(x) = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10(1 - \ln x).$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $\ln x = 1$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(e) = 10e$

Endpoint values: $f(1) = 10(2 - \ln 1) = 20$

$f(e^2) = 10e^2(2 - 2\ln e) = 0.$

We can see from this list that the function's absolute maximum value is $10e \approx 27.2$; it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$. ■

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

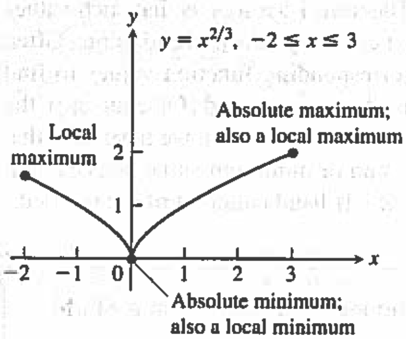


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

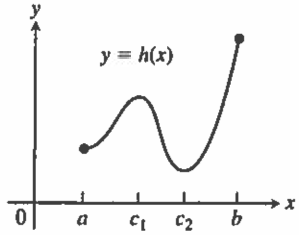
We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.9). ■

Exercises 4.1

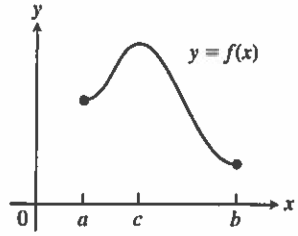
Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with .Theorem 1

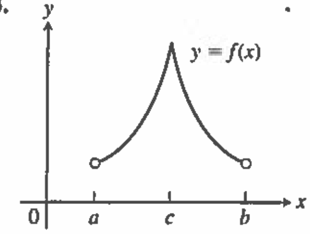
1.



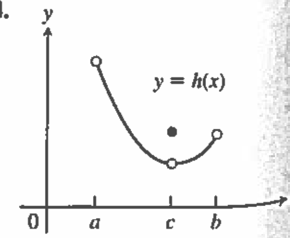
2.



3.



4.



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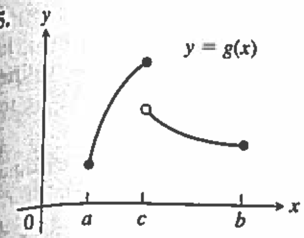

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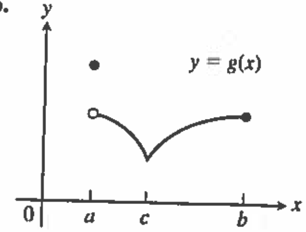
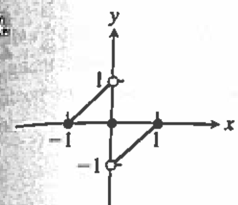
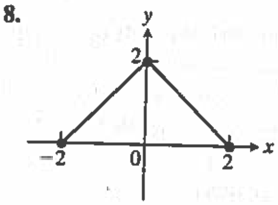


In Exercises 7–10, find the absolute extreme values and where they occur.

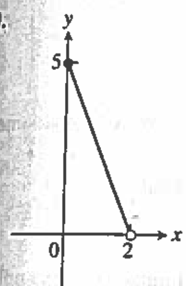
7.



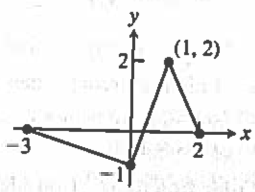
8.



9.



10.



In Exercises 11–14, match the table with a graph.

11.

x	$f'(x)$
a	0
b	0
c	5

12.

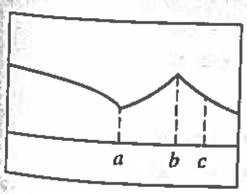
x	$f'(x)$
a	0
b	0
c	−5

13.

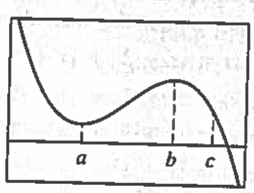
x	$f'(x)$
a	does not exist
b	0
c	−2

14.

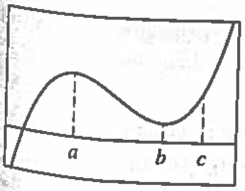
x	$f'(x)$
a	does not exist
b	does not exist
c	−1.7



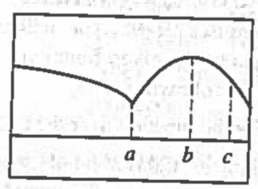
(a)



(b)



(c)



(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

15. $f(x) = |x|, \quad -1 < x < 2$

16. $y = \dfrac{6}{x^2 + 2}, \quad -1 < x < 1$

17. $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

18. $h(x) = \begin{cases} \dfrac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$

19. $y = 3 \sin x, \quad 0 < x < 2\pi$

20. $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ \cos x, & 0 < x \le \dfrac{\pi}{2} \end{cases}$

Absolute Extrema on Finite Closed Intervals

In Exercises 21–40, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

22. $f(x) = -x - 4, \quad -4 \le x \le 1$

23. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

24. $f(x) = 4 - x^3, \quad -2 \le x \le 1$

25. $F(x) = -\dfrac{1}{x^2}, \quad 0.5 \le x \le 2$

26. $F(x) = -\dfrac{1}{x}, \quad -2 \le x \le -1$

27. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

28. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

29. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

30. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

31. $f(\theta) = \sin \theta, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$

32. $f(\theta) = \tan \theta, \quad -\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{4}$

33. $g(x) = \csc x, \quad \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$

34. $g(x) = \sec x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$

35. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

36. $f(t) = |t - 5|, \quad 4 \le t \le 7$

37. $g(x) = xe^{-x}, \quad -1 \le x \le 1$

38. $h(x) = \ln(x + 1), \quad 0 \le x \le 3$

39. $f(x) = \dfrac{1}{x} + \ln x, \quad 0.5 \le x \le 4$

40. $g(x) = e^{-x^2}, \quad -2 \le x \le 1$

In Exercises 41–44, find the function's absolute maximum and minimum values and say where they are assumed.

41. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

42. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

43. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

44. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Finding Critical Points

In Exercises 45–52, determine all critical points for each function.

45. $y = x^2 - 6x + 7$

46. $f(x) = 6x^2 - x^3$

47. $f(x) = x(4 - x)^3$

48. $g(x) = (x - 1)^2(x - 3)^2$

49. $y = x^2 + \dfrac{2}{x}$

50. $f(x) = \dfrac{x^2}{x - 2}$

51. $y = x^2 - 32\sqrt{x}$

52. $g(x) = \sqrt{2x - x^2}$

Finding Extreme Values

In Exercises 53–68, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

53. $y = 2x^2 - 8x + 9$

54. $y = x^3 - 2x + 4$

55. $y = x^3 + x^2 - 8x + 5$

56. $y = x^3(x - 5)^2$

57. $y = \sqrt{x^2 - 1}$

58. $y = x - 4\sqrt{x}$

59. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

60. $y = \sqrt{3 + 2x - x^2}$

61. $y = \dfrac{x}{x^2 + 1}$

62. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

63. $y = e^x + e^{-x}$

64. $y = e^x - e^{-x}$

65. $y = x \ln x$

66. $y = x^2 \ln x$

67. $y = \cos^{-1}(x^2)$

68. $y = \sin^{-1}(e^x)$

Local Extrema and Critical Points

In Exercises 69–76, find the critical points, domain endpoints, and extreme values (absolute and local) for each function.

69. $y = x^{2/3}(x + 2)$

70. $y = x^{2/3}(x^2 - 4)$

71. $y = x\sqrt{4 - x^2}$

72. $y = x^2\sqrt{3 - x}$

73. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

74. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

75. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

76. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 77 and 78, give reasons for your answers.

77. Let $f(x) = (x - 2)^{2/3}$.

a. Does $f'(2)$ exist?

b. Show that the only local extreme value of f occurs at $x = 2$.

c. Does the result in part (b) contradict the Extreme Value Theorem?

d. Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

78. Let $f(x) = |x^3 - 9x|$.

a. Does $f'(0)$ exist?

b. Does $f'(3)$ exist?

c. Does $f'(-3)$ exist?

d. Determine all extrema of f.

Theory and Examples

79. A minimum with no derivative The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

80. Even functions If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

81. Odd functions If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

82. No critical points or endpoints exist We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

83. The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.

a. Find the extreme values of V.

b. Interpret any values found in part (a) in terms of the volume of the box.

84. Cubic functions Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

a. Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

b. How many local extreme values can f have?

85. Maximum height of a vertically moving body The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \qquad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

86. Peak alternating current Suppose that at any given time t (in seconds) the current i (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

T Graph the functions in Exercises 87–90. Then find the extreme values of the function on the interval and say where they occur.

87. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

88. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

89. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

90. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

COMPUTER EXPLORATIONS

In Exercises 91–98, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

a. Plot the function over the interval to see its general behavior there.

b. Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.

c. Find the interior points where f' does not exist.

d. Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

e. Find the function's absolute extreme values on the interval and identify where they occur.

91. $f(x) = x^4 - 8x^2 + 4x + 2, \quad [-20/25, 64/25]$

92. $f(x) = -x^4 + 4x^3 - 4x + 1, \quad [-3/4, 3]$

93. $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

94. $f(x) = 2 + 2x - 3x^{2/3}, \quad [-1, 10/3]$

95. $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

96. $f(x) = x^{3/4} - \sin x + \dfrac{1}{2}, \quad [0, 2\pi]$

97. $f(x) = \pi x^2 e^{-3x/2}, \quad [0, 5]$

98. $f(x) = \ln(2x + x\sin x), \quad [1, 15]$

4.2 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

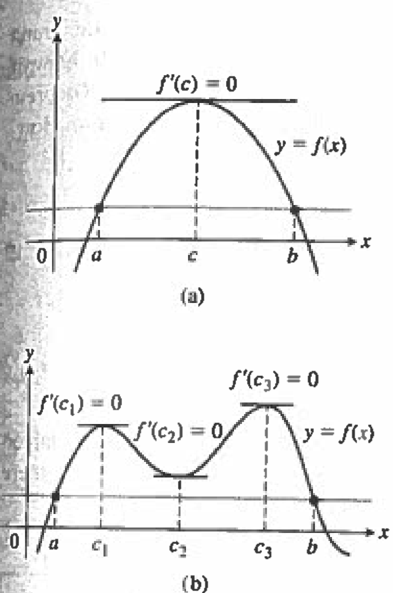


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.10). We now state and prove this result.

> **THEOREM 3—Rolle's Theorem** Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b). ■

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

HISTORICAL BIOGRAPHY
Michel Rolle
(1652–1719)

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Exercises 4.2

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–8.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$
5. $f(x) = \sin^{-1} x, \quad [-1, 1]$
6. $f(x) = \ln(x - 1), \quad [2, 4]$
7. $f(x) = x^3 - x^2, \quad [-1, 2]$
8. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 9–14 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

9. $f(x) = x^{2/3}, \quad [-1, 8]$
10. $f(x) = x^{4/5}, \quad [0, 1]$
11. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$
12. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
13. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$
14. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$
15. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

16. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

17. a. Plot the zeros of each polynomial on a line together with the zeros of its first derivative.
    i) $y = x^2 - 4$
    ii) $y = x^2 + 8x + 15$
    iii) $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$
    iv) $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

    b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

18. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.
19. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
20. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 21–28 have exactly one zero in the given interval.

21. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$
22. $f(x) = x^3 + \frac{4}{x^2} + 7, \quad (-\infty, 0)$
23. $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$
24. $g(t) = \frac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$
25. $r(\theta) = \theta + \sin^2\left(\frac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$
26. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$
27. $r(\theta) = \sec\theta - \frac{1}{\theta^3} + 5, \quad (0, \pi/2)$
28. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

Finding Functions from Derivatives

29. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.
30. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.
31. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if
    a. $f(0) = 0$ b. $f(1) = 0$ c. $f(-2) = 3$.
32. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 33–38, find all possible functions with the given derivative.

33. a. $y' = x$ b. $y' = x^2$ c. $y' = x^3$
34. a. $y' = 2x$ b. $y' = 2x - 1$ c. $y' = 3x^2 + 2x - 1$
35. a. $y' = -\frac{1}{x^2}$ b. $y' = 1 - \frac{1}{x^2}$ c. $y' = 5 + \frac{1}{x^2}$

36. a. $y' = \frac{1}{2\sqrt{x}}$ b. $y' = \frac{1}{\sqrt{x}}$ c. $y' = 4x - \frac{1}{\sqrt{x}}$

37. a. $y' = \sin 2t$ b. $y' = \cos \frac{t}{2}$ c. $y' = \sin 2t + \cos \frac{t}{2}$

38. a. $y' = \sec^2 \theta$ b. $y' = \sqrt{\theta}$ c. $y' = \sqrt{\theta} - \sec^2 \theta$

In Exercises 39–42, find the function with the given derivative whose graph passes through the point P.

39. $f'(x) = 2x - 1, \quad P(0, 0)$

40. $g'(x) = \frac{1}{x^2} + 2x, \quad P(-1, 1)$

41. $f'(x) = e^{2x}, \quad P\left(0, \frac{3}{2}\right)$

42. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

Finding Position from Velocity or Acceleration

Exercises 43–46 give the velocity $v = ds/dt$ and initial position of an object moving along a coordinate line. Find the object's position at time t.

43. $v = 9.8t + 5, \quad s(0) = 10$

44. $v = 32t - 2, \quad s(0.5) = 4$

45. $v = \sin \pi t, \quad s(0) = 0$

46. $v = \frac{2}{\pi} \cos \frac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 47–50 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of an object moving on a coordinate line. Find the object's position at time t.

47. $a = e^t, \quad v(0) = 20, \quad s(0) = 5$

48. $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

49. $a = -4 \sin 2t, \quad v(0) = 2, \quad s(0) = -3$

50. $a = \frac{9}{\pi^2} \cos \frac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

Applications

51. **Temperature change** It took 14 sec for a mercury thermometer to rise from -19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of 8.5°C/sec.

52. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

53. Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea or nautical miles per hour).

54. A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

55. Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

56. **Free fall on the moon** On our moon, the acceleration of gravity is 1.6 m/sec^2. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

Theory and Examples

57. **The geometric mean of a and b** The *geometric mean* of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

58. **The arithmetic mean of a and b** The *arithmetic mean* of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T 59. Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

60. **Rolle's Theorem**

a. Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1,$ and 2.

b. Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?

c. Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f'?

61. **Unique solution** Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

62. **Parallel tangents** Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

63. Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

64. Suppose that $0 < f'(x) < 1/2$ for all x-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

65. Show that $|\cos x - 1| \leq |x|$ for all x-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

66. Show that for any numbers a and b, the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.

67. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

68. If $|f(w) - f(x)| \leq |w - x|$ for all values w and x and f is a differentiable function, show that $-1 \leq f'(x) \leq 1$ for all x-values.

69. Assume that f is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

70. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answers.

T 71. Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

T 72. Use the inequalities in Exercise 70 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

73. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

a. Show that $f(x) \ge 1$ for all x.

b. Must $f'(1) = 0$? Explain.

74. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

75. Use the same-derivative argument, as was done to prove the Product and Power Rules for logarithms, to prove the Quotient Rule property.

76. Use the same-derivative argument to prove the identities

a. $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ b. $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2}$

77. Starting with the equation $e^{x_1}e^{x_2} = e^{x_1+x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1-x_2}$ for any numbers x_1 and x_2.

78. Show that $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ■

Corollary 3 tells us that $f(x) = \sqrt{x}$ is increasing on the interval $[0, b]$ for any $b > 0$ because $f'(x) = 1/\sqrt{x}$ is positive on $(0, b)$. The derivative does not exist at $x = 0$, but Corollary 3 still applies. The corollary is valid for infinite as well as finite intervals, so $f(x) = \sqrt{x}$ is increasing on $[0, \infty)$.

To find the intervals where a function f is increasing or decreasing, we first find all of the critical points of f. If $a < b$ are two critical points for f, and if the derivative f' is continuous but never zero on the interval (a, b), then by the Intermediate Value Theorem applied to f', the derivative must be everywhere positive on (a, b), or everywhere negative there. One way we can determine the sign of f' on (a, b) is simply by evaluating the derivative at a single point c in (a, b). If $f'(c) > 0$, then $f'(x) > 0$ for all x in (a, b) so f is increasing on $[a, b]$ by Corollary 3; if $f'(c) < 0$, then f is decreasing on $[a, b]$. The next example illustrates how we use this procedure.

EXAMPLE 3 Find the critical points of

$$f(x) = (x^2 - 3)e^x.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of f'.

Using the Derivative Product Rule, we find the derivative

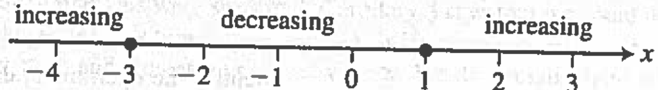
$$\begin{aligned} f'(x) &= (x^2 - 3)\cdot\frac{d}{dx}e^x + \frac{d}{dx}(x^2 - 3)\cdot e^x \\ &= (x^2 - 3)\cdot e^x + (2x)\cdot e^x \\ &= (x^2 + 2x - 3)e^x. \end{aligned}$$

Since e^x is never zero, the first derivative is zero if and only if

$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ (x + 3)(x - 1) &= 0. \end{aligned}$$

The zeros $x = -3$ and $x = 1$ partition the x-axis into open intervals as follows.

Interval	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



We can see from the table that there is a local maximum (about 0.299) at $x = -3$ and a local minimum (about -5.437) at $x = 1$. The local minimum value is also an absolute minimum because $f(x) > 0$ for $|x| > \sqrt{3}$. There is no absolute maximum. The function increases on $(-\infty, -3)$ and $(1, \infty)$ and decreases on $(-3, 1)$. Figure 4.23 shows the graph. ■

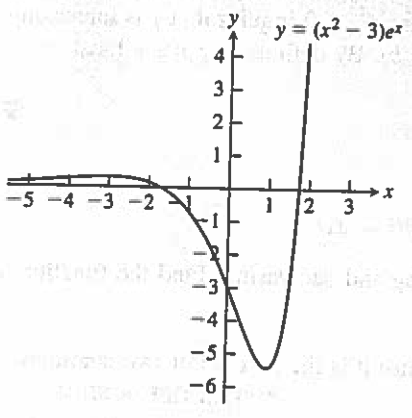


FIGURE 4.23 The graph of $f(x) = (x^2 - 3)e^x$ (Example 3).

Exercises 4.3

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what open intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$
2. $f'(x) = (x - 1)(x + 2)$
3. $f'(x) = (x - 1)^2(x + 2)$
4. $f'(x) = (x - 1)^2(x + 2)^2$
5. $f'(x) = (x - 1)e^{-x}$
6. $f'(x) = (x - 7)(x + 1)(x + 5)$
7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$
8. $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$
9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$
10. $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$
11. $f'(x) = x^{-1/3}(x + 2)$
12. $f'(x) = x^{-1/2}(x - 3)$
13. $f'(x) = (\sin x - 1)(2\cos x + 1), 0 \le x \le 2\pi$
14. $f'(x) = (\sin x + \cos x)(\sin x - \cos x), 0 \le x \le 2\pi$

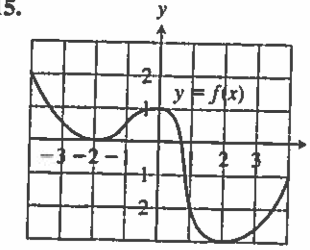
Identifying Extrema

In Exercises 15–44:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

15.



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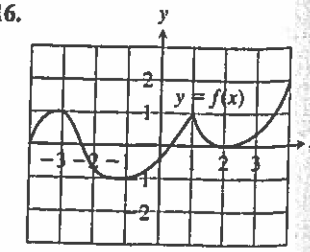

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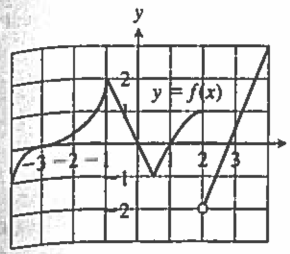

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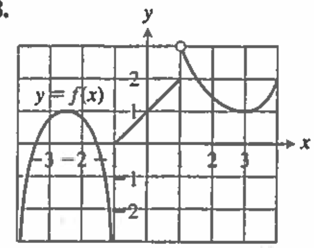


19. $g(t) = -t^2 - 3t + 3$

20. $g(t) = -3t^2 + 9t + 5$

21. $h(x) = -x^3 + 2x^2$

22. $h(x) = 2x^3 - 18x$

23. $f(\theta) = 3\theta^2 - 4\theta^3$

24. $f(\theta) = 6\theta - \theta^3$

25. $f(r) = 3r^3 + 16r$

26. $h(r) = (r + 7)^3$

27. $f(x) = x^4 - 8x^2 + 16$

28. $g(x) = x^4 - 4x^3 + 4x^2$

29. $H(t) = \frac{3}{2}t^4 - t^6$

30. $K(t) = 15t^3 - t^5$

31. $f(x) = x - 6\sqrt{x - 1}$

32. $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$

34. $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

36. $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x + 8)$

38. $g(x) = x^{2/3}(x + 5)$

39. $h(x) = x^{1/3}(x^2 - 4)$

40. $k(x) = x^{2/3}(x^2 - 4)$

41. $f(x) = e^{2x} + e^{-x}$

42. $f(x) = e^{\sqrt{x}}$

43. $f(x) = x \ln x$

44. $f(x) = x^2 \ln x$

In Exercises 45–56:

a. Identify the function's local extreme values in the given domain, and say where they occur.

b. Which of the extreme values, if any, are absolute?

T c. Support your findings with a graphing calculator or computer grapher.

45. $f(x) = 2x - x^2, \quad -\infty < x \leq 2$

46. $f(x) = (x + 1)^2, \quad -\infty < x \leq 0$

47. $g(x) = x^2 - 4x + 4, \quad 1 \leq x < \infty$

48. $g(x) = -x^2 - 6x - 9, \quad -4 \leq x < \infty$

49. $f(t) = 12t - t^3, \quad -3 \leq t < \infty$

50. $f(t) = t^3 - 3t^2, \quad -\infty < t \leq 3$

51. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \leq x < \infty$

52. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \leq 0$

53. $f(x) = \sqrt{25 - x^2}, \quad -5 \leq x \leq 5$

54. $f(x) = \sqrt{x^2 - 2x - 3}, \quad 3 \leq x < \infty$

55. $g(x) = \dfrac{x - 2}{x^2 - 1}, \quad 0 \leq x < 1$

56. $g(x) = \dfrac{x^2}{4 - x^2}, \quad -2 < x \leq 1$

In Exercises 57–64:

a. Find the local extrema of each function on the given interval, and say where they occur.

T b. Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

57. $f(x) = \sin 2x, \quad 0 \leq x \leq \pi$

58. $f(x) = \sin x - \cos x, \quad 0 \leq x \leq 2\pi$

59. $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \leq x \leq 2\pi$

60. $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

61. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \leq x \leq 2\pi$

62. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \leq x \leq \pi$

63. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

64. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

Theory and Examples

Show that the functions in Exercises 65 and 66 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

65. $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \leq \theta \leq 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

66. $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \leq \theta \leq \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

67. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

a. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

b. $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

c. $f'(x) > 0$ for $x \neq 1$;

d. $f'(x) < 0$ for $x \neq 1$.

68. Sketch the graph of a differentiable function $y = f(x)$ that has

a. a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

b. a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

c. local maxima at $(1, 1)$ and $(3, 3)$;

d. local minima at $(1, 1)$ and $(3, 3)$.

69. Sketch the graph of a continuous function $y = g(x)$ such that

a. $g(2) = 2, 0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

b. $g(2) = 2, g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

70. Sketch the graph of a continuous function $y = h(x)$ such that

a. $h(0) = 0, -2 \leq h(x) \leq 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

b. $h(0) = 0, -2 \leq h(x) \leq 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

71. Discuss the extreme-value behavior of the function $f(x) = x\sin(1/x)$, $x \neq 0$. How many critical points does this function have? Where are they located on the x-axis? Does f have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

72. Find the open intervals on which the function $f(x) = ax^2 + bx + c$, $a \neq 0$, is increasing and decreasing. Describe the reasoning behind your answer.

73. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

74. Determine the values of constants a, b, c, and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

75. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

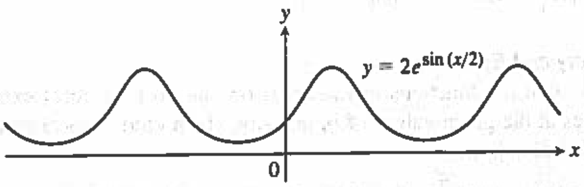
b. $\cos(\ln x)$ on $[1/2, 2]$.

76. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

77. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

78. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



79. Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

80. a. Prove that $e^x \ge 1 + x$ if $x \ge 0$.

b. Use the result in part (a) to show that

$$e^x \ge 1 + x + \frac{1}{2}x^2.$$

81. Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 81 to show that the functions in Exercises 82–86 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 3, Section 3.8.

82. $f(x) = (1/3)x + (5/6)$

83. $f(x) = 27x^3$

84. $f(x) = 1 - 8x^3$

85. $f(x) = (1 - x)^3$

86. $f(x) = x^{5/3}$

4.4 Concavity and Curve Sketching

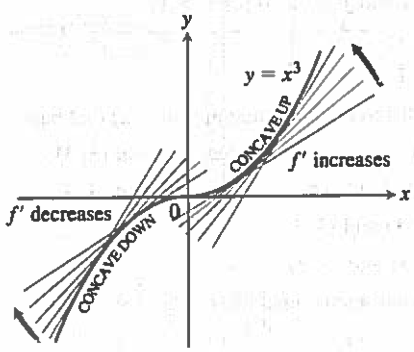


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior studied in Sections 1.1 and 2.6, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data.

Concavity

As you can see in Figure 4.24, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

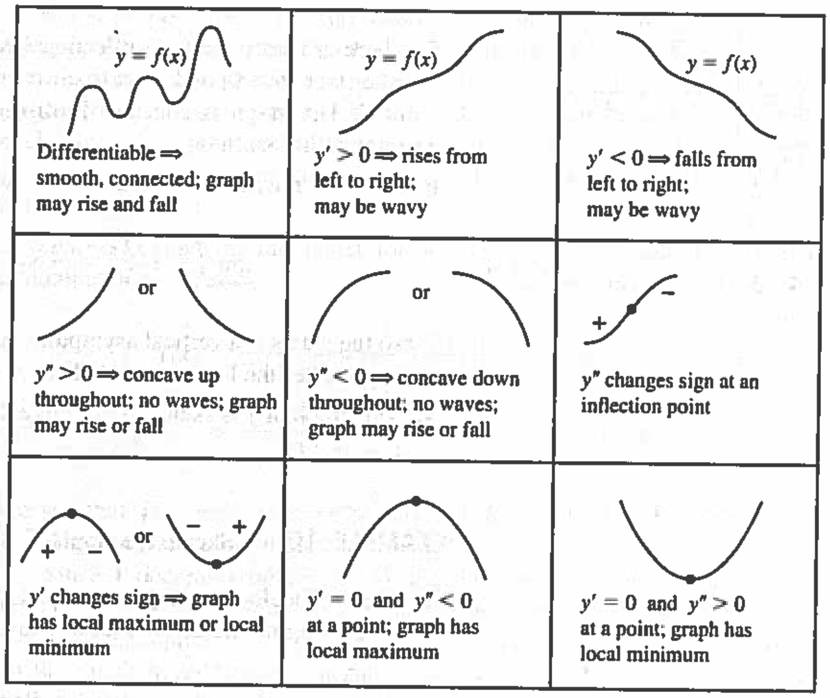
DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that f' increases if $f'' > 0$ on I, and decreases if $f'' < 0$.

figure summarizes how the first derivative and second derivative affect the shape of a graph.

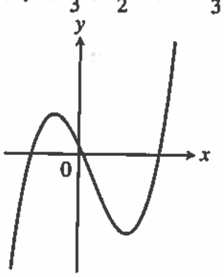


Exercises 4.4

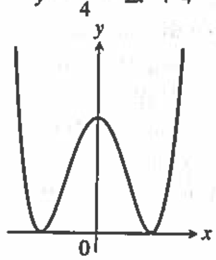
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

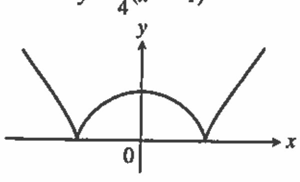
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



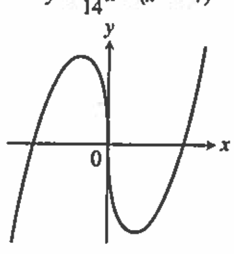
2. $y = \frac{x^4}{4} - 2x^2 + 4$



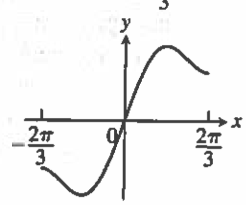
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



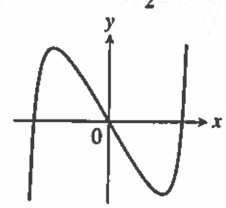
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



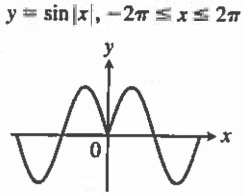
5. $y = x + \sin 2x,\ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



6. $y = \tan x - 4x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

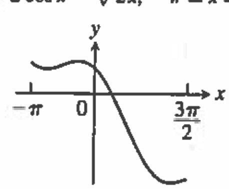


7. $y = \sin|x|,\ -2\pi \le x \le 2\pi$



NOT TO SCALE

8. $y = 2\cos x - \sqrt{2}x,\ -\pi \le x \le \frac{3\pi}{2}$



Graphing Functions

In Exercises 9–58, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

9. $y = x^2 - 4x + 3$
10. $y = 6 - 2x - x^2$
11. $y = x^3 - 3x + 3$
12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

32. $y = \dfrac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$

34. $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\frac{5}{2} - x\right)$

36. $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$

38. $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$

40. $y = x^2 + \frac{2}{x}$

41. $y = \dfrac{x^2 - 3}{x - 2}$

42. $y = \sqrt[3]{x^3 + 1}$

43. $y = \dfrac{8x}{x^2 + 4}$

44. $y = \dfrac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$

46. $y = |x^2 - 2x|$

47. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

48. $y = \sqrt{|x - 4|}$

49. $y = xe^{1/x}$

50. $y = \dfrac{e^x}{x}$

51. $y = \ln(3 - x^2)$

52. $y = x(\ln x)^2$

53. $y = e^x - 2e^{-x} - 3x$

54. $y = xe^{-x}$

55. $y = \ln(\cos x)$

56. $y = \dfrac{\ln x}{\sqrt{x}}$

57. $y = \dfrac{1}{1 + e^{-x}}$

58. $y = \dfrac{e^x}{1 + e^x}$

Sketching the General Shape, Knowing y'

Each of Exercises 59–80 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure on page 249 to sketch the general shape of the graph of f.

59. $y' = 2 + x - x^2$

60. $y' = x^2 - x - 6$

61. $y' = x(x - 3)^2$

62. $y' = x^2(2 - x)$

63. $y' = x(x^2 - 12)$

64. $y' = (x - 1)^2(2x + 3)$

65. $y' = (8x - 5x^2)(4 - x)^2$

66. $y' = (x^2 - 2x)(x - 5)^2$

67. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

68. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

69. $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

70. $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

71. $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

72. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

73. $y' = \cos t, \quad 0 \le t \le 2\pi$

74. $y' = \sin t, \quad 0 \le t \le 2\pi$

75. $y' = (x + 1)^{-2/3}$

76. $y' = (x - 2)^{-1/3}$

77. $y' = x^{-2/3}(x - 1)$

78. $y' = x^{-4/5}(x + 1)$

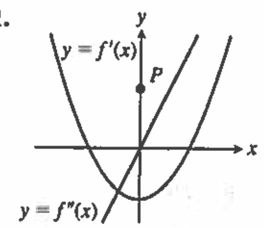
79. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

80. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

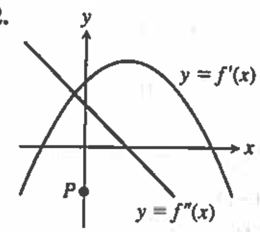
Sketching y from Graphs of y' and y''

Each of Exercises 81–84 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

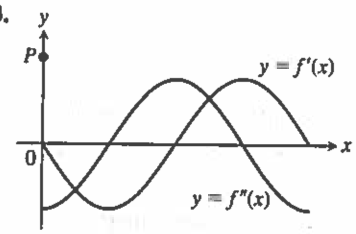
81.



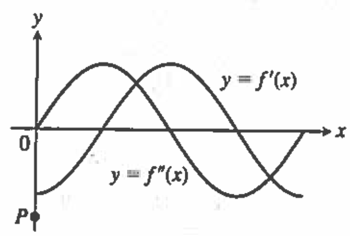
82.



83.



84.



Graphing Rational Functions

Graph the rational functions in Exercises 85–102 using all the steps in the graphing procedure on page 249.

85. $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

86. $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

87. $y = \dfrac{x^4 + 1}{x^2}$

88. $y = \dfrac{x^2 - 4}{2x}$

89. $y = \dfrac{1}{x^2 - 1}$

90. $y = \dfrac{x^2}{x^2 - 1}$

91. $y = -\dfrac{x^2 - 2}{x^2 - 1}$

92. $y = \dfrac{x^2 - 4}{x^2 - 2}$

93. $y = \dfrac{x^2}{x + 1}$

94. $y = -\dfrac{x^2 - 4}{x + 1}$

95. $y = \dfrac{x^2 - x + 1}{x - 1}$

96. $y = -\dfrac{x^2 - x + 1}{x - 1}$

97. $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

98. $y = \dfrac{x^3 + x - 2}{x - x^2}$

99. $y = \dfrac{x}{x^2 - 1}$

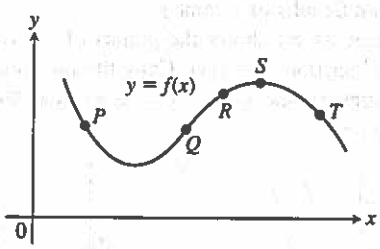
100. $y = \dfrac{x - 1}{x^2(x - 2)}$

101. $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

102. $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

Theory and Examples

103. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



104. Sketch a smooth connected curve $y = f(x)$ with

$$\begin{array}{ll} f(-2) = 8, & f'(2) = f'(-2) = 0, \\ f(0) = 4, & f'(x) < 0 \quad \text{for} \quad |x| < 2, \\ f(2) = 0, & f''(x) < 0 \quad \text{for} \quad x < 0, \\ f'(x) > 0 \quad \text{for} \quad |x| > 2, & f''(x) > 0 \quad \text{for} \quad x > 0. \end{array}$$

105. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0,\ y'' > 0$
2	1	$y' = 0,\ y'' > 0$
$2 < x < 4$		$y' > 0,\ y'' > 0$
4	4	$y' > 0,\ y'' = 0$
$4 < x < 6$		$y' > 0,\ y'' < 0$
6	7	$y' = 0,\ y'' < 0$
$x > 6$		$y' < 0,\ y'' < 0$

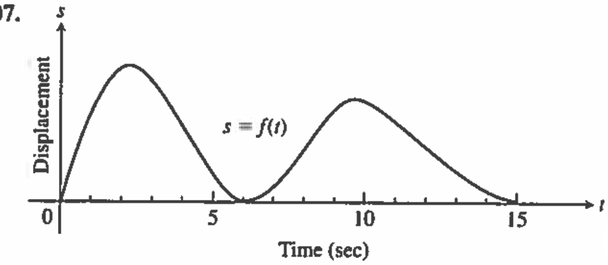
106. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$$y': \quad \underset{}{+} \;\; \underset{-2}{|} \;\; - \;\; \underset{0}{|} \;\; + \;\; \underset{2}{|} \;\; -$$

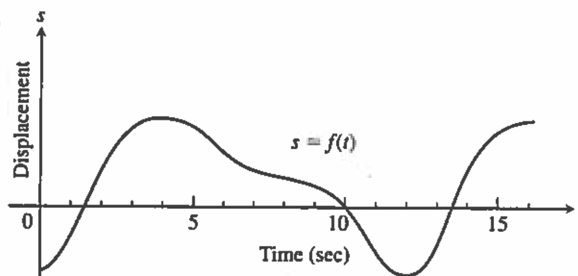
$$y'': \quad - \;\; \underset{-1}{|} \;\; + \;\; \underset{1}{|} \;\; -$$

Motion Along a Line The graphs in Exercises 107 and 108 show the position $s = f(t)$ of an object moving up and down on a coordinate line. (a) When is the object moving away from the origin? Toward the origin? At approximately what times is the (b) velocity equal to zero? (c) Acceleration equal to zero? (d) When is the acceleration positive? Negative?

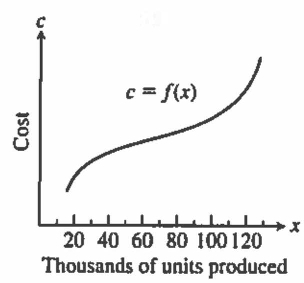
107.



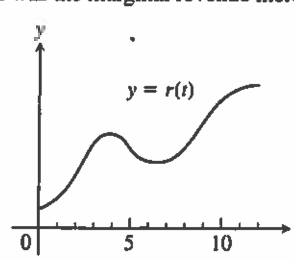
108.



109. Marginal cost The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



110. The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?



111. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

112. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x-1)^2(x-2)(x-4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

113. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

114. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

115. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

116. **Parabolas**

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

117. **Quadratic curves** What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

118. **Cubic curves** What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

119. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x+1)(x-2).$$

For what x-values does the graph of f have an inflection point?

120. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x-2)^3(x+3).$$

For what x-values does the graph of f have an inflection point?

121. Find the values of constants a, b, and c so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

122. Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

COMPUTER EXPLORATIONS

In Exercises 123–126, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

123. $y = x^5 - 5x^4 - 240$

124. $y = x^3 - 12x^2$

125. $y = \frac{4}{5}x^5 + 16x^2 - 25$

126. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

127. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

128. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

4.5 Indeterminate Forms and L'Hôpital's Rule

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital (1661–1704)

Johann Bernoulli (1667–1748)

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule,** after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

Exercises 4.5

Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to-2}\dfrac{x+2}{x^2-4}$

2. $\lim_{x\to0}\dfrac{\sin 5x}{x}$

3. $\lim_{x\to\infty}\dfrac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to1}\dfrac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to0}\dfrac{1-\cos x}{x^2}$

6. $\lim_{x\to\infty}\dfrac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to2}\dfrac{x-2}{x^2-4}$

8. $\lim_{x\to-5}\dfrac{x^2-25}{x+5}$

9. $\lim_{t\to-3}\dfrac{t^3-4t+15}{t^2-t-12}$

10. $\lim_{t\to-1}\dfrac{3t^3+3}{4t^3-t+3}$

11. $\lim_{x\to\infty}\dfrac{5x^3-2x}{7x^3+3}$

12. $\lim_{x\to\infty}\dfrac{x-8x^2}{12x^2+5x}$

13. $\lim_{t\to0}\dfrac{\sin t^2}{t}$

14. $\lim_{t\to0}\dfrac{\sin 5t}{2t}$

15. $\lim_{x\to0}\dfrac{8x^2}{\cos x-1}$

16. $\lim_{x\to0}\dfrac{\sin x-x}{x^3}$

17. $\lim_{\theta\to\pi/2}\dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$

18. $\lim_{\theta\to-\pi/3}\dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

19. $\lim_{\theta\to\pi/2}\dfrac{1-\sin\theta}{1+\cos 2\theta}$

20. $\lim_{x\to1}\dfrac{x-1}{\ln x-\sin \pi x}$

21. $\lim_{x\to0}\dfrac{x^2}{\ln(\sec x)}$

22. $\lim_{x\to\pi/2}\dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$

23. $\lim_{t\to0}\dfrac{t(1-\cos t)}{t-\sin t}$

24. $\lim_{t\to0}\dfrac{t\sin t}{1-\cos t}$

25. $\lim_{x\to(\pi/2)^-}\left(x-\dfrac{\pi}{2}\right)\sec x$

26. $\lim_{x\to(\pi/2)^-}\left(\dfrac{\pi}{2}-x\right)\tan x$

27. $\lim_{\theta\to0}\dfrac{3^{\sin\theta}-1}{\theta}$

28. $\lim_{\theta\to0}\dfrac{(1/2)^{\theta}-1}{\theta}$

29. $\lim_{x\to0}\dfrac{x2^x}{2^x-1}$

30. $\lim_{x\to0}\dfrac{3^x-1}{2^x-1}$

31. $\lim_{x\to\infty}\dfrac{\ln(x+1)}{\log_2 x}$

32. $\lim_{x\to\infty}\dfrac{\log_2 x}{\log_3(x+3)}$

33. $\lim_{x\to0^+}\dfrac{\ln(x^2+2x)}{\ln x}$

34. $\lim_{x\to0^+}\dfrac{\ln(e^x-1)}{\ln x}$

35. $\lim_{y\to0}\dfrac{\sqrt{5y+25}-5}{y}$

36. $\lim_{y\to0}\dfrac{\sqrt{ay+a^2}-a}{y}, \quad a>0$

37. $\lim_{x\to\infty}(\ln 2x-\ln(x+1))$

38. $\lim_{x\to0^+}(\ln x-\ln\sin x)$

39. $\lim_{x\to0^+}\dfrac{(\ln x)^2}{\ln(\sin x)}$

40. $\lim_{x\to0^+}\left(\dfrac{3x+1}{x}-\dfrac{1}{\sin x}\right)$

41. $\lim_{x\to1^+}\left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$

42. $\lim_{x\to0^+}(\csc x-\cot x+\cos x)$

43. $\lim_{\theta\to0}\dfrac{\cos\theta-1}{e^{\theta}-\theta-1}$

44. $\lim_{h\to0}\dfrac{e^h-(1+h)}{h^2}$

45. $\lim_{t\to\infty}\dfrac{e^t+t^2}{e^t-t}$

46. $\lim_{x\to\infty}x^2e^{-x}$

47. $\lim_{x\to0}\dfrac{x-\sin x}{x\tan x}$

48. $\lim_{x\to0}\dfrac{(e^x-1)^2}{x\sin x}$

49. $\lim_{\theta\to0}\dfrac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$

50. $\lim_{x\to0}\dfrac{\sin 3x-3x+x^2}{\sin x\sin 2x}$

Indeterminate Powers and Products

Find the limits in Exercises 51–66.

51. $\lim_{x\to1^+}x^{1/(1-x)}$

52. $\lim_{x\to1^+}x^{1/(x-1)}$

53. $\lim_{x\to\infty}(\ln x)^{1/x}$

54. $\lim_{x\to e^+}(\ln x)^{1/(x-e)}$

55. $\lim_{x\to0^+}x^{-1/\ln x}$

56. $\lim_{x\to\infty}x^{1/\ln x}$

57. $\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}$

58. $\lim_{x\to0}(e^x+x)^{1/x}$

59. $\lim_{x\to0^+}x^x$

60. $\lim_{x\to0^+}\left(1+\dfrac{1}{x}\right)^x$

61. $\lim_{x\to\infty}\left(\dfrac{x+2}{x-1}\right)^x$

62. $\lim_{x\to\infty}\left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

63. $\lim_{x\to0^+}x^2\ln x$

64. $\lim_{x\to0^+}x(\ln x)^2$

65. $\lim_{x\to0^+}x\tan\left(\dfrac{\pi}{2}-x\right)$

66. $\lim_{x\to0^+}\sin x\cdot\ln x$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

67. $\lim_{x\to\infty}\dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$

68. $\lim_{x\to0^+}\dfrac{\sqrt{x}}{\sqrt{\sin x}}$

69. $\lim_{x\to(\pi/2)^-}\dfrac{\sec x}{\tan x}$

70. $\lim_{x\to0^+}\dfrac{\cot x}{\csc x}$

71. $\lim_{x\to\infty}\dfrac{2^x-3^x}{3^x+4^x}$

72. $\lim_{x\to-\infty}\dfrac{2^x+4^x}{5^x-2^x}$

73. $\lim_{x\to\infty}\dfrac{e^{x^2}}{xe^x}$

74. $\lim_{x\to0^+}\dfrac{x}{e^{-1/x}}$

75. Which one is correct, and which one is wrong? Give reasons for your answers.

a. $\lim_{x\to3}\dfrac{x-3}{x^2-3}=\lim_{x\to3}\dfrac{1}{2x}=\dfrac{1}{6}$ b. $\lim_{x\to3}\dfrac{x-3}{x^2-3}=\dfrac{0}{6}=0$

76. Which one is correct, and which one is wrong? Give reasons for your answers.

a. $\lim_{x\to0}\dfrac{x^2-2x}{x^2-\sin x}=\lim_{x\to0}\dfrac{2x-2}{2x-\cos x}$

$=\lim_{x\to0}\dfrac{2}{2+\sin x}=\dfrac{2}{2+0}=1$

b. $\lim_{x\to0}\dfrac{x^2-2x}{x^2-\sin x}=\lim_{x\to0}\dfrac{2x-2}{2x-\cos x}=\dfrac{-2}{0-1}=2$

77. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a. $\lim_{x\to0^+} x\ln x = 0\cdot(-\infty) = 0$

b. $\lim_{x\to0^+} x\ln x = 0\cdot(-\infty) = -\infty$

c. $\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{(1/x)} = \frac{-\infty}{\infty} = -1$

d. $\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{(1/x)}$

$= \lim_{x\to0^+}\frac{(1/x)}{(-1/x^2)} = \lim_{x\to0^+}(-x) = 0$

78. Find all values of c that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x,\quad g(x) = x^2,\quad (a, b) = (-2, 0)$

b. $f(x) = x,\quad g(x) = x^2,\quad (a, b)$ arbitrary

c. $f(x) = x^3/3 - 4x,\quad g(x) = x^2,\quad (a, b) = (0, 3)$

79. **Continuous extension** Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

80. For what values of a and b is

$$\lim_{x\to0}\left(\frac{\tan 2x}{x^3} + \frac{a}{x^2} + \frac{\sin bx}{x}\right) = 0?$$

T 81. **$\infty - \infty$ Form**

a. Estimate the value of

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + x}\right)$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of x-values.

b. Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $\left(x + \sqrt{x^2 + x}\right)/\left(x + \sqrt{x^2 + x}\right)$ and simplify the new numerator.

82. Find $\lim_{x\to\infty}\left(\sqrt{x^2 + 1} - \sqrt{x}\right)$.

T 83. **0/0 Form** Estimate the value of

$$\lim_{x\to1}\frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

84. This exercise explores the difference between the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

a. Use l'Hôpital's Rule to show that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

T b. Graph

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \quad\text{and}\quad g(x) = \left(1 + \frac{1}{x}\right)^x$$

together for $x \ge 0$. How does the behavior of f compare with that of g? Estimate the value of $\lim_{x\to\infty} f(x)$.

c. Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's Rule.

85. Show that

$$\lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^k = e^r.$$

86. Given that $x > 0$, find the maximum value, if any, of

a. $x^{1/x}$

b. x^{1/x^2}

c. x^{1/x^n} (n a positive integer)

d. Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer n.

87. Use limits to find horizontal asymptotes for each function.

a. $y = x\tan\left(\frac{1}{x}\right)$ b. $y = \dfrac{3x + e^{2x}}{2x + e^{3x}}$

88. Find $f'(0)$ for $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

T 89. **The continuous extension of $(\sin x)^x$ to $[0, \pi]$**

a. Graph $f(x) = (\sin x)^x$ on the interval $0 \le x \le \pi$. What value would you assign to f to make it continuous at $x = 0$?

b. Verify your conclusion in part (a) by finding $\lim_{x\to0^+} f(x)$ with l'Hôpital's Rule.

c. Returning to the graph, estimate the maximum value of f on $[0, \pi]$. About where is max f taken on?

d. Sharpen your estimate in part (c) by graphing f' in the same window to see where its graph crosses the x-axis. To simplify your work, you might want to delete the exponential factor from the expression for f' and graph just the factor that has a zero.

T 90. **The function $(\sin x)^{\tan x}$** *(Continuation of Exercise 89.)*

a. Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \le x \le 7$. How do you account for the gaps in the graph? How wide are the gaps?

b. Now graph f on the interval $0 \le x \le \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for f at $x = \pi/2$? (*Hint:* Use l'Hôpital's Rule to find lim f as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

c. Continuing with the graphs in part (b), find max f and min f as accurately as you can and estimate the values of x at which they are taken on.