

Of the two critical points, only  $\sqrt{200}$  lies in the domain of  $c(x)$ . The critical point value of the average daily cost is

$$c(\sqrt{200}) = \frac{5000}{\sqrt{200}} + 25\sqrt{200} = 500\sqrt{2} \approx \$707.11.$$

We note that  $c(x)$  is defined over the open interval  $(0, \infty)$  with  $c''(x) = 10000/x^3 > 0$ . Thus, an absolute minimum exists at  $x = \sqrt{200} \approx 14.14$  days.

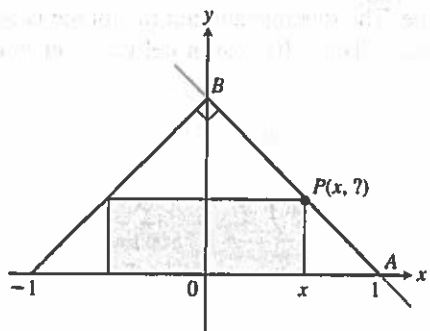
The cabinetmaker should schedule a delivery of  $5(14) = 70$  units of the mahogany wood every 14 days.

## Exercises 4.6

### Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

- Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is  $16 \text{ in}^2$ , and what are its dimensions?
- Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.
- The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
  - Express the  $y$ -coordinate of  $P$  in terms of  $x$ . (*Hint:* Write an equation for the line  $AB$ .)
  - Express the area of the rectangle in terms of  $x$ .
  - What is the largest area the rectangle can have, and what are its dimensions?



- A rectangle has its base on the  $x$ -axis and its upper two vertices on the parabola  $y = 12 - x^2$ . What is the largest area the rectangle can have, and what are its dimensions?
- You are planning to make an open rectangular box from an 8-in.-by-15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?
- You are planning to close off a corner of the first quadrant with a line segment 20 units long running from  $(a, 0)$  to  $(0, b)$ . Show that the area of the triangle enclosed by the segment is largest when  $a = b$ .
- The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a

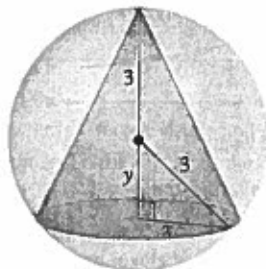
single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

- The shortest fence** A  $216 \text{ m}^2$  rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?
- Designing a tank** Your iron works has contracted to design and build a  $500 \text{ ft}^3$ , square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.
  - What dimensions do you tell the shop to use?
  - Briefly describe how you took weight into account.
- Catching rainwater** A  $1125 \text{ ft}^3$  open-top rectangular tank with a square base  $x$  ft on a side and  $y$  ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product  $xy$ .
  - If the total cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of  $x$  and  $y$  will minimize it?

- Give a possible scenario for the cost function in part (a).
- Designing a poster** You are designing a rectangular poster to contain  $50 \text{ in}^2$  of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?
  - Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



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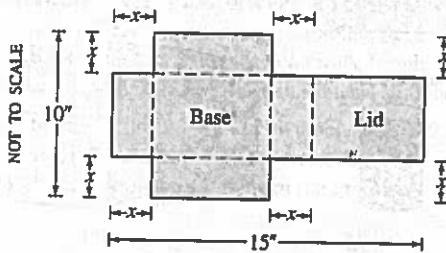
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13. Two sides of a triangle have lengths  $a$  and  $b$ , and the angle between them is  $\theta$ . What value of  $\theta$  will maximize the triangle's area? (Hint:  $A = (1/2)ab\sin\theta$ .)
14. **Designing a can** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of  $1000 \text{ cm}^3$ ? Compare the result here with the result in Example 2.
15. **Designing a can** You are designing a  $1000 \text{ cm}^3$  right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius  $r$  will be cut from squares that measure  $2r$  units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the  $A = 2\pi r^2 + 2\pi rh$  in Example 2. In Example 2, the ratio of  $h$  to  $r$  for the most economical can was 2 to 1. What is the ratio now?

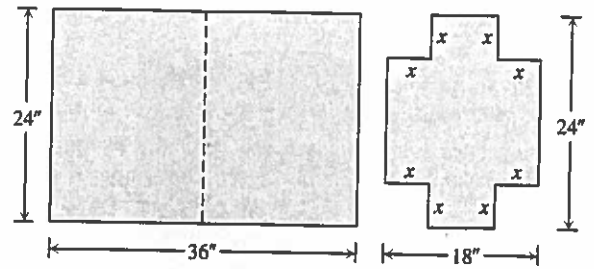
16. **Designing a box with a lid** A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



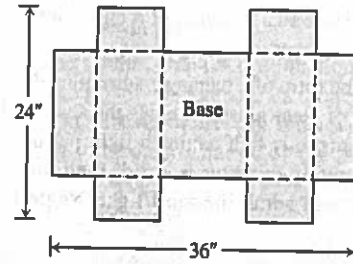
- Write a formula  $V(x)$  for the volume of the box.
- Find the domain of  $V$  for the problem situation and graph  $V$  over this domain.
- Use a graphical method to find the maximum volume and the value of  $x$  that gives it.
- Confirm your result in part (c) analytically.

17. **Designing a suitcase** A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length  $x$  are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

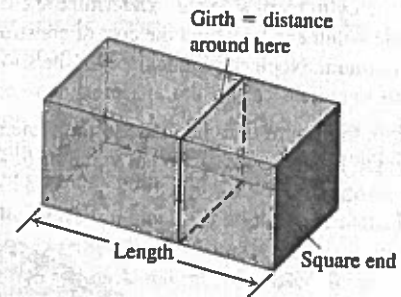
- Write a formula  $V(x)$  for the volume of the box.
- Find the domain of  $V$  for the problem situation and graph  $V$  over this domain.
- Use a graphical method to find the maximum volume and the value of  $x$  that gives it.
- Confirm your result in part (c) analytically.
- Find a value of  $x$  that yields a volume of  $1120 \text{ in}^3$ .
- Write a paragraph describing the issues that arise in part (b).



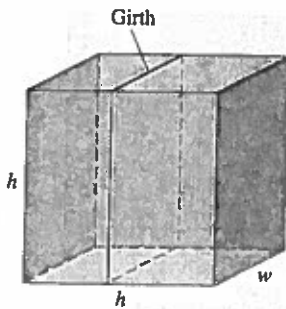
The sheet is then unfolded.



- A rectangle is to be inscribed under the arch of the curve  $y = 4 \cos(0.5x)$  from  $x = -\pi$  to  $x = \pi$ . What are the dimensions of the rectangle with largest area, and what is the largest area?
- Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?
- a. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?

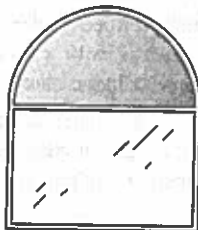


- Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length and compare what you see with your answer in part (a).
21. (Continuation of Exercise 20.)
- Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are  $h$  by  $h$  by  $w$  and the girth is  $2h + 2w$ . What dimensions will give the box its largest volume now?

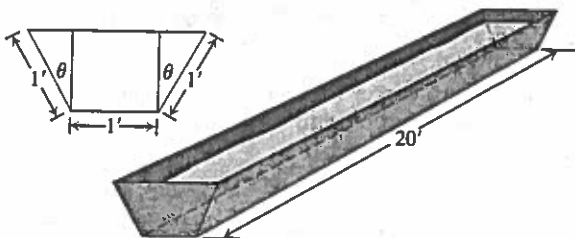


**T** b. Graph the volume as a function of  $h$  and compare what you see with your answer in part (a).

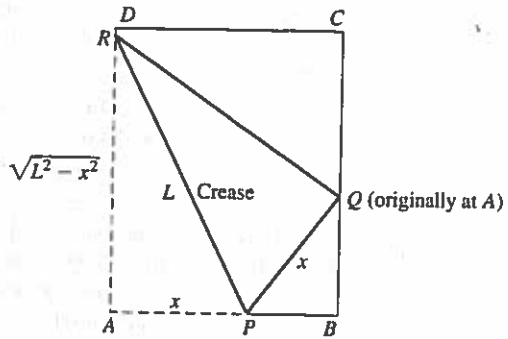
22. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



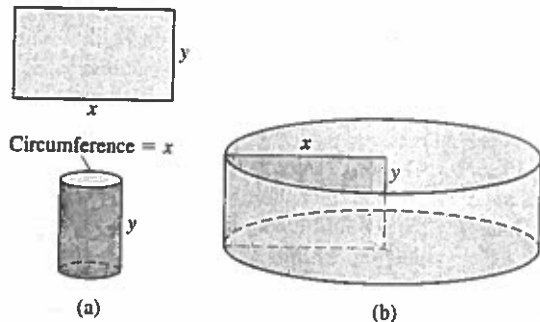
23. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.
24. The trough in the figure is to be made to the dimensions shown. Only the angle  $\theta$  can be varied. What value of  $\theta$  will maximize the trough's volume?



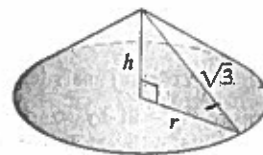
25. **Paper folding** A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length  $L$ . Try it with paper.
- Show that  $L^2 = 2x^3/(2x - 8.5)$ .
  - What value of  $x$  minimizes  $L^2$ ?
  - What is the minimum value of  $L$ ?



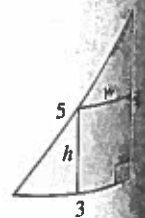
26. **Constructing cylinders** Compare the answers to the following two construction problems.
- A rectangular sheet of perimeter 36 cm and dimensions  $x$  cm by  $y$  cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of  $x$  and  $y$  give the largest volume?
  - The same sheet is to be revolved about one of the sides of length  $y$  to sweep out the cylinder as shown in part (b) of the figure. What values of  $x$  and  $y$  give the largest volume?



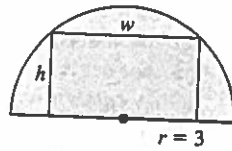
27. **Constructing cones** A right triangle whose hypotenuse is  $\sqrt{3}$  m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



- Find the point on the line  $\frac{x}{a} + \frac{y}{b} = 1$  that is closest to the origin.
- Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.
- Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.
- A wire  $b$  m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?
- Answer Exercise 31 if one piece is bent into a square and the other into a circle.
- Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.



34. Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See accompanying figure.)



35. What value of  $a$  makes  $f(x) = x^2 + (a/x)$  have
- a local minimum at  $x = 2$ ?
  - a point of inflection at  $x = 1$ ?
36. What values of  $a$  and  $b$  make  $f(x) = x^3 + ax^2 + bx$  have
- a local maximum at  $x = -1$  and a local minimum at  $x = 3$ ?
  - a local minimum at  $x = 4$  and a point of inflection at  $x = 1$ ?

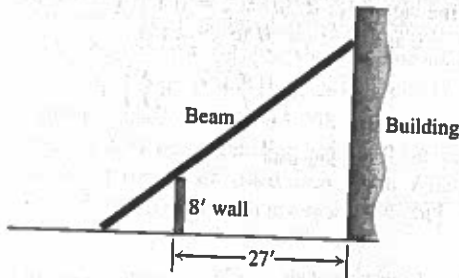
**Physical Applications**

37. **Vertical motion** The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with  $s$  in feet and  $t$  in seconds. Find

- the object's velocity when  $t = 0$ ;
  - its maximum height and when it occurs;
  - its velocity when  $s = 0$ .
38. **Quickest route** Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?
39. **Shortest beam** The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

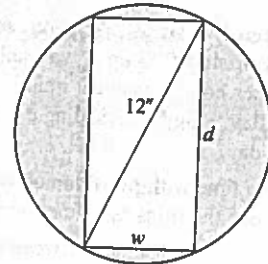


40. **Motion on a line** The positions of two particles on the  $s$ -axis are  $s_1 = \sin t$  and  $s_2 = \sin(t + \pi/3)$ , with  $s_1$  and  $s_2$  in meters and  $t$  in seconds.
- At what time(s) in the interval  $0 \leq t \leq 2\pi$  do the particles meet?
  - What is the farthest apart that the particles ever get?
  - When in the interval  $0 \leq t \leq 2\pi$  is the distance between the particles changing the fastest?
41. The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?
42. **Projectile motion** The range  $R$  of a projectile fired from the origin over horizontal ground is the distance from the origin to the point of impact. If the projectile is fired with an initial velocity  $v_0$  at an angle  $\alpha$  with the horizontal, then in Chapter 13 we find that

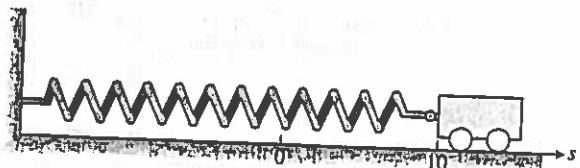
$$R = \frac{v_0^2}{g} \sin 2\alpha,$$

where  $g$  is the downward acceleration due to gravity. Find the angle  $\alpha$  for which the range  $R$  is the largest possible.

- T** 43. **Strength of a beam** The strength  $S$  of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)
- Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.
  - Graph  $S$  as a function of the beam's width  $w$ , assuming the proportionality constant to be  $k = 1$ . Reconcile what you see with your answer in part (a).
  - On the same screen, graph  $S$  as a function of the beam's depth  $d$ , again taking  $k = 1$ . Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of  $k$ ? Try it.

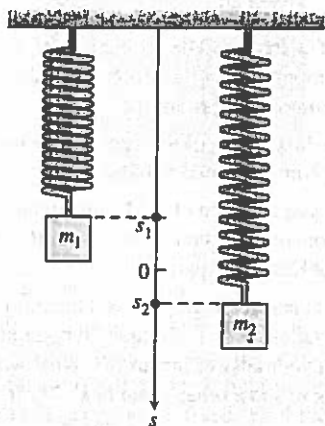


- T** 44. **Stiffness of a beam** The stiffness  $S$  of a rectangular beam is proportional to its width times the cube of its depth.
- Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.
  - Graph  $S$  as a function of the beam's width  $w$ , assuming the proportionality constant to be  $k = 1$ . Reconcile what you see with your answer in part (a).
  - On the same screen, graph  $S$  as a function of the beam's depth  $d$ , again taking  $k = 1$ . Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of  $k$ ? Try it.
45. **Frictionless cart** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time  $t = 0$  to roll back and forth for 4 sec. Its position at time  $t$  is  $s = 10 \cos \pi t$ .
- What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?
  - Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



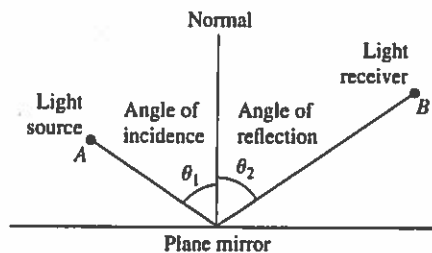
46. Two masses hanging side by side from springs have positions  $s_1 = 2 \sin t$  and  $s_2 = \sin 2t$ , respectively.
- At what times in the interval  $0 < t$  do the masses pass each other? (Hint:  $\sin 2t = 2 \sin t \cos t$ .)

- b. When in the interval  $0 \leq t \leq 2\pi$  is the vertical distance between the masses the greatest? What is this distance? (Hint:  $\cos 2t = 2\cos^2 t - 1$ .)



47. **Distance between two ships** At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour; a nautical mile is 2000 yd) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.
- Start counting time with  $t = 0$  at noon and express the distance  $s$  between the ships as a function of  $t$ .
  - How rapidly was the distance between the ships changing at noon? One hour later?
  - The visibility that day was 5 nautical miles. Did the ships ever sight each other?
- T** d. Graph  $s$  and  $ds/dt$  together as functions of  $t$  for  $-1 \leq t \leq 3$ , using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).
- e. The graph of  $ds/dt$  looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that  $ds/dt$  approaches a limiting value as  $t \rightarrow \infty$ . What is this value? What is its relation to the ships' individual speeds?

48. **Fermat's principle in optics** Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



49. **Tin pest** When metallic tin is kept below  $13.2^\circ\text{C}$ , it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their

churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate  $v = dx/dt$  of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is,  $v$  may be considered to be a function of  $x$  alone, and

$$v = kx(a - x) = kax - kx^2,$$

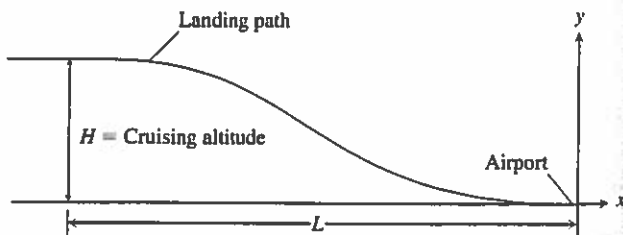
where

- $x$  = the amount of product
- $a$  = the amount of substance at the beginning
- $k$  = a positive constant.

At what value of  $x$  does the rate  $v$  have a maximum? What is the maximum value of  $v$ ?

50. **Airplane landing path** An airplane is flying at altitude  $H$  when it begins its descent to an airport runway that is at horizontal ground distance  $L$  from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function  $y = ax^3 + bx^2 + cx + d$ , where  $y(-L) = H$  and  $y(0) = 0$ .
- What is  $dy/dx$  at  $x = 0$ ?
  - What is  $dy/dx$  at  $x = -L$ ?
  - Use the values for  $dy/dx$  at  $x = 0$  and  $x = -L$  together with  $y(0) = 0$  and  $y(-L) = H$  to show that

$$y(x) = H \left[ 2 \left( \frac{x}{L} \right)^3 + 3 \left( \frac{x}{L} \right)^2 \right].$$



**Business and Economics**

51. It costs you  $c$  dollars each to manufacture and distribute backpacks. If the backpacks sell at  $x$  dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where  $a$  and  $b$  are positive constants. What selling price will bring a maximum profit?

52. You operate a tour service that offers the following rates:  
 \$200 per person if 50 people (the minimum number to book the tour) go on the tour.  
 For each additional person, up to a maximum of 80 people total, the rate per person is reduced by \$2.  
 It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

53. **Wilson lot size formula** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where  $q$  is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be),  $k$  is the cost of placing an order (the same, no matter how often you order),  $c$  is the cost of one item (a constant),  $m$  is the number of items sold each week (a constant), and  $h$  is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

- a. Your job, as the inventory manager for your store, is to find the quantity that will minimize  $A(q)$ . What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)
- b. Shipping costs sometimes depend on order size. When they do, it is more realistic to replace  $k$  by  $k + bq$ , the sum of  $k$  and a constant multiple of  $q$ . What is the most economical quantity to order now?
54. **Production level** Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.
55. Show that if  $r(x) = 6x$  and  $c(x) = x^3 - 6x^2 + 15x$  are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).
56. **Production level** Suppose that  $c(x) = x^3 - 20x^2 + 20,000x$  is the cost of manufacturing  $x$  items. Find a production level that will minimize the average cost of making  $x$  items.
57. You are to construct an open rectangular box with a square base and a volume of  $48 \text{ ft}^3$ . If material for the bottom costs  $\$6/\text{ft}^2$  and material for the sides costs  $\$4/\text{ft}^2$ , what dimensions will result in the least expensive box? What is the minimum cost?
58. The 800-room Mega Motel chain is filled to capacity when the room charge is  $\$50$  per night. For each  $\$10$  increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

#### Biology

59. **Sensitivity to medicine** (Continuation of Exercise 72, Section 3.3.) Find the amount of medicine to which the body is most sensitive by finding the value of  $M$  that maximizes the derivative  $dR/dM$ , where

$$R = M^2 \left( \frac{C}{2} - \frac{M}{3} \right)$$

and  $C$  is a constant.

#### 60. How we cough

- a. When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity  $v$  can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \quad \frac{r_0}{2} \leq r \leq r_0,$$

where  $r_0$  is the rest radius of the trachea in centimeters and  $c$  is a positive constant whose value depends in part on the length of the trachea.

Show that  $v$  is greatest when  $r = (2/3)r_0$ ; that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

- T** b. Take  $r_0$  to be 0.5 and  $c$  to be 1 and graph  $v$  over the interval  $0 \leq r \leq 0.5$ . Compare what you see with the claim that  $v$  is at a maximum when  $r = (2/3)r_0$ .

#### Theory and Examples

61. **An inequality for positive integers** Show that if  $a, b, c$ , and  $d$  are positive integers, then

$$\frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd} \geq 16.$$

62. **The derivative  $dt/dx$  in Example 4**

- a. Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of  $x$ .

- b. Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

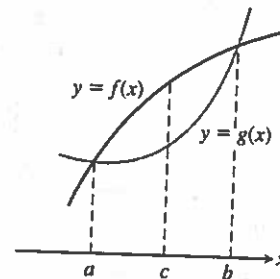
is a decreasing function of  $x$ .

- c. Show that

$$\frac{dt}{dx} = \frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d - x}{c_2 \sqrt{b^2 + (d - x)^2}}$$

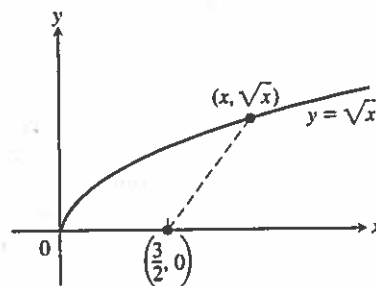
is an increasing function of  $x$ .

63. Let  $f(x)$  and  $g(x)$  be the differentiable functions graphed here. Point  $c$  is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at  $c$ ? Give reasons for your answer.



64. You have been asked to determine whether the function  $f(x) = 3 + 4 \cos x + \cos 2x$  is ever negative.
- Explain why you need to consider values of  $x$  only in the interval  $[0, 2\pi]$ .
  - Is  $f$  ever negative? Explain.
65. a. The function  $y = \cot x - \sqrt{2} \csc x$  has an absolute maximum value on the interval  $0 < x < \pi$ . Find it.
- T** b. Graph the function and compare what you see with your answer in part (a).
66. a. The function  $y = \tan x + 3 \cot x$  has an absolute minimum value on the interval  $0 < x < \pi/2$ . Find it.
- T** b. Graph the function and compare what you see with your answer in part (a).
67. a. How close does the curve  $y = \sqrt{x}$  come to the point  $(3/2, 0)$ ? (*Hint: If you minimize the square of the distance, you can avoid square roots.*)

- T** b. Graph the distance function  $D(x)$  and  $y = \sqrt{x}$  together and reconcile what you see with your answer in part (a).



68. a. How close does the semicircle  $y = \sqrt{16 - x^2}$  come to the point  $(1, \sqrt{3})$ ?
- T** b. Graph the distance function and  $y = \sqrt{16 - x^2}$  together and reconcile what you see with your answer in part (a).

## 4.7 Newton's Method

In this section we study a numerical method, called *Newton's method* or the *Newton-Raphson method*, which is a technique to approximate the solution to an equation  $f(x) = 0$ . Essentially it uses tangent lines of the graph of  $y = f(x)$  near the points where  $f$  is zero to estimate the solution. (A value of  $x$  where  $f$  is zero is a *root* of the function  $f$  and a *solution* of the equation  $f(x) = 0$ .)

### Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation  $f(x) = 0$  is to produce a sequence of approximations that approach the solution. We pick the first number  $x_0$  of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of  $f$  crosses the  $x$ -axis (Figure 4.46). At each step the method approximates a zero of  $f$  with a zero of one of its linearizations. Here is how it works.

The initial estimate,  $x_0$ , may be found by graphing or just plain guessing. The method then uses the tangent to the curve  $y = f(x)$  at  $(x_0, f(x_0))$  to approximate the curve, calling the point  $x_1$  where the tangent meets the  $x$ -axis (Figure 4.46). The number  $x_1$  is usually a better approximation to the solution than is  $x_0$ . The point  $x_2$  where the tangent to the curve at  $(x_1, f(x_1))$  crosses the  $x$ -axis is the next approximation in the sequence. We continue on using each approximation to generate the next, until we are close enough to the root to stop.

We can derive a formula for generating the successive approximations in the following way. Given the approximation  $x_n$ , the point-slope equation for the tangent to the curve at  $(x_n, f(x_n))$  is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

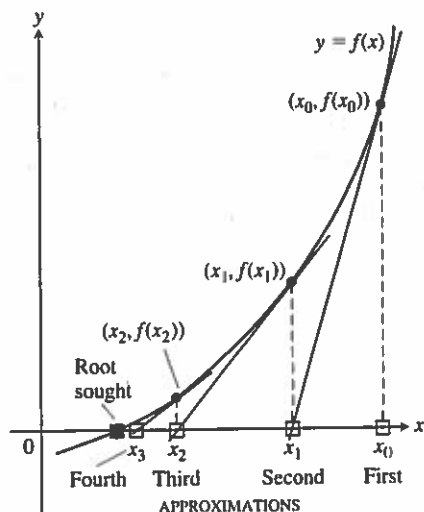
We can find where it crosses the  $x$ -axis by setting  $y = 0$  (Figure 4.47):

$$0 = f(x_n) + f'(x_n)(x - x_n)$$

$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{if } f'(x_n) \neq 0$$

This value of  $x$  is the next approximation  $x_{n+1}$ . Here is a summary of Newton's method.



**FIGURE 4.46** Newton's method starts with an initial guess  $x_0$  and (under favorable circumstances) improves the guess one step at a time.

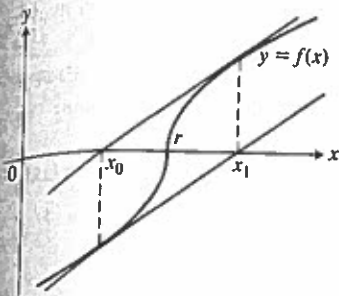


FIGURE 4.51 Newton's method fails to converge. You go from  $x_0$  to  $x_1$  and back to  $x_0$ , never getting any closer to  $r$ .

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.52 shows two ways this can happen.

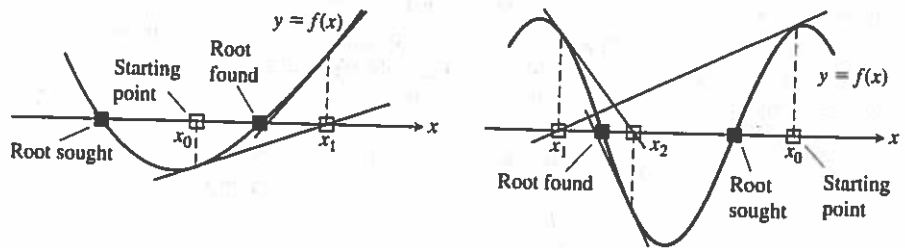


FIGURE 4.52 If you start too far away, Newton's method may miss the root you want.

## Exercises 4.7

### Root Finding

- Use Newton's method to estimate the solutions of the equation  $x^2 + x - 1 = 0$ . Start with  $x_0 = -1$  for the left-hand solution and with  $x_0 = 1$  for the solution on the right. Then, in each case, find  $x_2$ .
- Use Newton's method to estimate the one real solution of  $x^3 + 3x + 1 = 0$ . Start with  $x_0 = 0$  and then find  $x_2$ .
- Use Newton's method to estimate the two zeros of the function  $f(x) = x^4 + x - 3$ . Start with  $x_0 = -1$  for the left-hand zero and with  $x_0 = 1$  for the zero on the right. Then, in each case, find  $x_2$ .
- Use Newton's method to estimate the two zeros of the function  $f(x) = 2x - x^2 + 1$ . Start with  $x_0 = 0$  for the left-hand zero and with  $x_0 = 2$  for the zero on the right. Then, in each case, find  $x_2$ .
- Use Newton's method to find the positive fourth root of 2 by solving the equation  $x^4 - 2 = 0$ . Start with  $x_0 = 1$  and find  $x_2$ .
- Use Newton's method to find the negative fourth root of 2 by solving the equation  $x^4 - 2 = 0$ . Start with  $x_0 = -1$  and find  $x_2$ .
- Guessing a root** Suppose that your first guess is lucky, in the sense that  $x_0$  is a root of  $f(x) = 0$ . Assuming that  $f'(x_0)$  is defined and not 0, what happens to  $x_1$  and later approximations?
- Estimating pi** You plan to estimate  $\pi/2$  to five decimal places by using Newton's method to solve the equation  $\cos x = 0$ . Does it matter what your starting value is? Give reasons for your answer.

### Theory and Examples

9. **Oscillation** Show that if  $h > 0$ , applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to  $x_1 = -h$  if  $x_0 = h$  and to  $x_1 = h$  if  $x_0 = -h$ . Draw a picture that shows what is going on.

10. **Approximations that get worse and worse** Apply Newton's method to  $f(x) = x^{1/3}$  with  $x_0 = 1$  and calculate  $x_1, x_2, x_3$ , and  $x_4$ . Find a formula for  $|x_n|$ . What happens to  $|x_n|$  as  $n \rightarrow \infty$ ? Draw a picture that shows what is going on.

11. Explain why the following four statements ask for the same information:

- Find the roots of  $f(x) = x^3 - 3x - 1$ .
- Find the  $x$ -coordinates of the intersections of the curve  $y = x^3$  with the line  $y = 3x + 1$ .
- Find the  $x$ -coordinates of the points where the curve  $y = x^3 - 3x$  crosses the horizontal line  $y = 1$ .
- Find the values of  $x$  where the derivative of  $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$  equals zero.

12. **Locating a planet** To calculate a planet's space coordinates, we have to solve equations like  $x = 1 + 0.5 \sin x$ . Graphing the function  $f(x) = x - 1 - 0.5 \sin x$  suggests that the function has a root near  $x = 1.5$ . Use one application of Newton's method to improve this estimate. That is, start with  $x_0 = 1.5$  and find  $x_1$ . (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

- T 13. **Intersecting curves** The curve  $y = \tan x$  crosses the line  $y = 2x$  between  $x = 0$  and  $x = \pi/2$ . Use Newton's method to find where.

- T 14. **Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation  $x^4 - 2x^3 - x^2 - 2x + 2 = 0$ .

- T 15. a. How many solutions does the equation  $\sin 3x = 0.99 - x^2$  have?

b. Use Newton's method to find them.

16. **Intersection of curves**

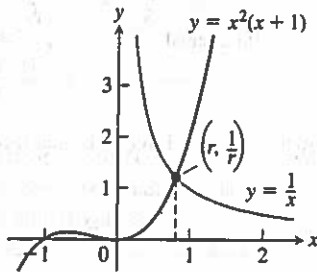
a. Does  $\cos 3x$  ever equal  $x$ ? Give reasons for your answer.

b. Use Newton's method to find where.

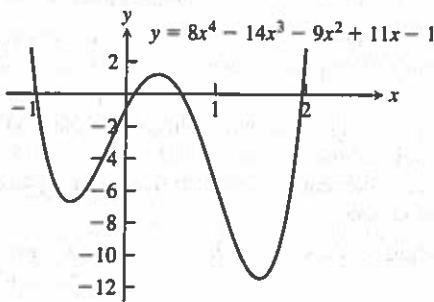
17. Find the four real zeros of the function  $f(x) = 2x^4 - 4x^2 + 1$ .



- T** 18. **Estimating pi** Estimate  $\pi$  to as many decimal places as your calculator will display by using Newton's method to solve the equation  $\tan x = 0$  with  $x_0 = 3$ .
19. **Intersection of curves** At what value(s) of  $x$  does  $\cos x = 2x$ ?
20. **Intersection of curves** At what value(s) of  $x$  does  $\cos x = -x$ ?
21. The graphs of  $y = x^2(x + 1)$  and  $y = 1/x$  ( $x > 0$ ) intersect at one point  $x = r$ . Use Newton's method to estimate the value of  $r$  to four decimal places.



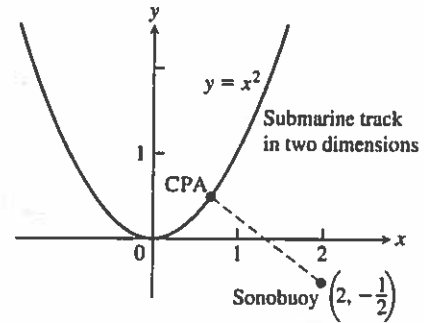
22. The graphs of  $y = \sqrt{x}$  and  $y = 3 - x^2$  intersect at one point  $x = r$ . Use Newton's method to estimate the value of  $r$  to four decimal places.
23. **Intersection of curves** At what value(s) of  $x$  does  $e^{-x^2} = x^2 - x + 1$ ?
24. **Intersection of curves** At what value(s) of  $x$  does  $\ln(1 - x^2) = x - 1$ ?
25. Use the Intermediate Value Theorem from Section 2.5 to show that  $f(x) = x^3 + 2x - 4$  has a root between  $x = 1$  and  $x = 2$ . Then find the root to five decimal places.
26. **Factoring a quartic** Find the approximate values of  $r_1$  through  $r_4$  in the factorization
- $$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



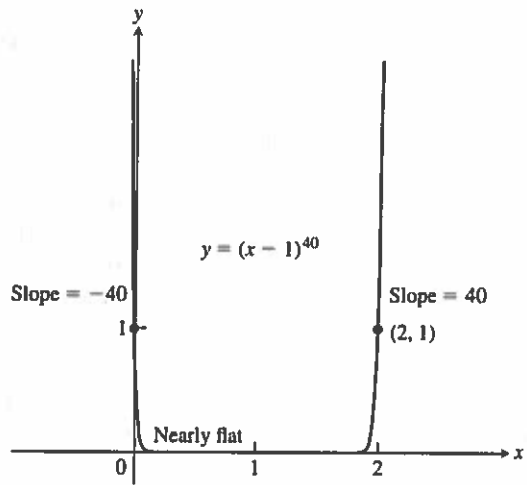
- T** 27. **Converging to different zeros** Use Newton's method to find the zeros of  $f(x) = 4x^4 - 4x^2$  using the given starting values.
- $x_0 = -2$  and  $x_0 = -0.8$ , lying in  $(-\infty, -\sqrt{2}/2)$
  - $x_0 = -0.5$  and  $x_0 = 0.25$ , lying in  $(-\sqrt{21}/7, \sqrt{21}/7)$
  - $x_0 = 0.8$  and  $x_0 = 2$ , lying in  $(\sqrt{2}/2, \infty)$
  - $x_0 = -\sqrt{21}/7$  and  $x_0 = \sqrt{21}/7$
28. **The sonobuoy problem** In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that

the submarine travels on the parabolic path  $y = x^2$  and that the buoy is located at the point  $(2, -1/2)$ .

- Show that the value of  $x$  that minimizes the distance between the submarine and the buoy is a solution of the equation  $x = 1/(x^2 + 1)$ .
- Solve the equation  $x = 1/(x^2 + 1)$  with Newton's method.



- T** 29. **Curves that are nearly flat at the root** Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on  $f(x) = (x - 1)^{40}$  with a starting value of  $x_0 = 2$  to see how close your machine comes to the root  $x = 1$ . See the accompanying graph.



30. The accompanying figure shows a circle of radius  $r$  with a chord of length 2 and an arc  $s$  of length 3. Use Newton's method to solve for  $r$  and  $\theta$  (radians) to four decimal places. Assume  $0 < \theta < \pi$ .

