RECALL: If function \( y = f(x) \) (graph passes vertical line test) is one-to-one (graph passes the horizontal line test), then \( f \) has an inverse function \( y = f^{-1}(x) \) satisfying

\[
f(f^{-1}(x)) = x.
\]

Notation: If \( f(a) = b \), then \( f^{-1}(b) = a \).

Example: (opt. prac.) Consider function \( f(x) = \frac{x}{x-2} \) and note that \( f(4) = 2 \).

1. Verify algebraically that \( f \) is 1-1, i.e., if \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

2. Find the inverse function \( y = f^{-1}(x) \).

3. Compute \( Df^{-1}(2) \).

Note: Sometimes finding \( y = f^{-1}(x) \) is difficult or impossible. Is there a way to find \( Df^{-1}(x) \) without first finding \( y = f^{-1}(x) \)?
**New Method**: If \( f(f^{-1}(x)) = x \), then by Chain Rule

\[
\begin{align*}
D \{ f(f^{-1}(x)) \} & = D \{ x^3 \} \\
\Rightarrow \quad f'(f^{-1}(x)) \cdot D f^{-1}(x) & = 1 \\
\Rightarrow \quad D f^{-1}(x) & = \frac{1}{f'(f^{-1}(x))}
\end{align*}
\]

**Example**: Consider function

\( f(x) = x^5 + x^3 + x + 5 \) and note that \( f(-1) = 2 \).

1.) Show that \( f \) is \((1,1)\) by using a derivative:

\[
\begin{align*}
f'(x) & = 5x^4 + 3x^2 + 1 > 0 \text{ for all } x \text{-values} \\
\Rightarrow \quad f \text{ is } \uparrow \quad \Rightarrow \quad f \text{ is } (1,1).
\end{align*}
\]

2.) Find \( D f^{-1}(2) \):

We have \( f^{-1}(2) = -1 \) and

\[
D f^{-1}(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(-1)}
\]

\[
= \frac{1}{5(-1)^4 + 3(-1)^2 + 1} = \frac{1}{8}
\]


Example: (opt. prac.) Consider function \( f(x) = e^{-x} - 2x^3 + 5 \) and note that \( f(0) = 6 \).

1.) Show that \( f \) is 1-1 by using a derivative.

2.) Find \( Df^{-1}(6) \).

Example: Assume that function

\[ f(x) = \frac{25}{34} \left[ (x-34)^3 + (x-34) \right] + 50 \]

for \( 30 \leq x \leq 34 \) is a mathematical model representing the number of "male" eggs in a clutch of 50 alligator eggs at temperature \( x \) (°C).

1.) Show that \( f \) is 1-1 and therefore has an inverse function \( y = f^{-1}(x) \):

\[ D \rightarrow f^{-1}(x) = \frac{25}{34} \left[ 3(x-34)^2 + 1 \right] > 0 \]

so \( f \) is \( \uparrow \rightarrow f \) is 1-1 \( \rightarrow f^{-1} \) exists.
Note that
\[ f(30) = 0 \text{ eggs, so } f^{-1}(0) = 30 \degree \text{C} \]
\[ f(31) \approx 28 \text{ eggs, so } f^{-1}(28) \approx 31 \degree \text{C} \]
\[ f(32) \approx 42 \text{ eggs, so } f^{-1}(42) \approx 32 \degree \text{C} \]
\[ f(33) \approx 48 \text{ eggs, so } f^{-1}(48) \approx 33 \degree \text{C} \]
\[ f(34) = 50 \text{ eggs, so } f^{-1}(50) = 34 \degree \text{C} \]

the units for \( Df(a) \) are \( \frac{\text{eggs}}{\degree \text{C}} \),

the units for \( Df^{-1}(b) \) are \( \frac{\degree \text{C}}{\text{egg}} \).

2.) Find \( Df(31) \):
\[
Df(x) = \frac{25}{34} \left[ 3(x - 34)^2 + 1 \right] \rightarrow \\
Df(31) = \frac{25}{34} \left[ 3(31 - 34)^2 + 1 \right] \approx 20.6 \frac{\text{eggs}}{\degree \text{C}}
\]

3.) Find \( Df^{-1}(28) \): Note \( f(31) = 28 \) and
\[
\rightarrow Df^{-1}(28) = \frac{1}{f'(f^{-1}(28))} \quad f^{-1}(28) = 31
\]
\[
= \frac{1}{f'(31)}
\]
\[
\approx \frac{1}{20.6} \approx 0.05 \frac{\degree \text{C}}{\text{egg}}
\]

4.) (opt. prac.) Find \( Df(33) \) and \( Df^{-1}(48) \).

5.) (opt. prac.) Find \( Df(34) \) and \( Df^{-1}(50) \).