Section 3.11

1.) $f(x) = x^3 - 2x + 3$, $a = 2 \xrightarrow{D}$
$f'(x) = 3x^2 - 2$ so linearization is
$L(x) = f(2) + f'(2)(x-2)$
$= 7 + 10(x-2)$
$= 7 + 10x - 20 = 10x - 13$

3.) $f(x) = x + \frac{1}{x}$, $a = 1 \xrightarrow{D} f'(x) = 1 - \frac{1}{x^2}$
so linearization is
$L(x) = f(1) + f'(1)(x-1)$
$= 2 + (0)(x-1) = 2$

4.) $f(x) = x^{1/3}$, $a = -8 \xrightarrow{D} f'(x) = \frac{1}{3}x^{-2/3}$
so linearization is
$L(x) = f(-8) + f'(-8)(x-(-8))$
$= (-8)^{1/3} + \frac{1}{3}(-8)^{-2/3}(x+8)$

$= -2 + \frac{1}{3}\cdot\frac{1}{4}(x+8)$
$= -2 + \frac{1}{12}x + \frac{2}{3} = \frac{1}{12}x - \frac{4}{3}$

5.) $f(x) = \tan x$, $a = \pi \xrightarrow{D} f'(x) = \sec^2 x$
so linearization is
$L(x) = f(\pi) + f'(\pi)(x-\pi)$
$= \tan\pi + \sec^2\pi\ (x-\pi)$
$= 0 + (-1)^2(x-\pi) = x - \pi$

6.) a.) $f(x) = \sin x$, $a = 0 \xrightarrow{D} f'(x) = \cos x$
so linearization is

$$L(x) = f(0) + f'(0)(x-0)$$
$$= \sin 0 + \cos 0 \cdot (x)$$
$$= 0 + (1)x = x$$

b.) $f(x) = \cos x$, $a=0 \xrightarrow{D} f'(x) = -\sin x$
so linearization is
$$L(x) = f(0) + f'(0)(x-0)$$
$$= \cos 0 + {}^{-}\sin 0 \cdot (x)$$
$$= 1 + (0)x = 1$$

c.) $f(x) = \tan x$, $a=0 \xrightarrow{D} f'(x) = \sec^2 x$
so linearization is
$$L(x) = f(0) + f'(0)(x-0)$$
$$= \tan 0 + \sec^2 0 \cdot (x)$$
$$= 0 + (1)^2 x = x$$

d.) $f(x) = e^x$, $a=0 \xrightarrow{D} f'(x) = e^x$
so linearization is
$$L(x) = f(0) + f'(0)(x-0)$$
$$= e^0 + e^0 (x) = 1 + (1)(x) = 1 + x$$

e.) $f(x) = \ln(1+x)$, $a=0 \xrightarrow{D} f'(x) = \frac{1}{1+x}$
so linearization is
$$L(x) = f(0) + f'(0)(x-0)$$
$$= \ln 1 + (1)(x)$$
$$= 0 + x = x$$

15.) $f(x) = (1+x)^k, \quad a=0 \xrightarrow{D}$

$f'(x) = k(1+x)^{k-1}$ so

linearization is

$L(x) = f(0) + f'(0)(x-0)$

$= (1)^k + k(1)^{k-1}(x)$

$= 1 + kx$

16.) $L(x) = \underline{1 + kx \approx (1+x)^k}$:

a.) $f(x) = (1-x)^6 = (1+(-x))^6$

$\approx 1 + 6(-x) = 1 - 6x$

c.) $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$

$\approx 1 + \left(-\frac{1}{2}\right)x = 1 - \frac{1}{2}x$

d.) $f(x) = \sqrt{2+x^2} = \sqrt{2\left(1+\frac{x^2}{2}\right)} = \sqrt{2}\sqrt{1+\frac{x^2}{2}}$

$= \sqrt{2}\cdot\left(1+\left(\frac{x^2}{2}\right)\right)^{1/2}$

$\approx \sqrt{2}\left(1+\frac{1}{2}\left(\frac{x^2}{2}\right)\right) = \sqrt{2}\left(1+\frac{1}{4}x^2\right)$

17.) a.) $(1+x)^k \approx 1+kx$ ~~so~~ $(1.0002)^{50} = (1+0.0002)^{50}$

$\approx 1 + 50(0.0002) = 1 + 0.01 = 1.01$

40.) $f(x) = 2x^2 + 4x - 3$, $x_0 = -1$, $dx = 0.1$

a.) $\Delta f = f(-0.9) - f(-1)$
$= -4.98 - (-5) = 0.02$

b.) $df = f'(-1)\,dx \quad (\xrightarrow{D} f'(x) = 4x + 4)$
$= (0)(0.1) = 0$

41.) $f(x) = x^3 - x$, $x_0 = 1$, $dx = 0.1$

a.) $\Delta f = f(1.1) - f(1)$
$= 0.231 - 0 = 0.231$

b.) $df = f'(1) \cdot dx \quad (\xrightarrow{D} f'(x) = 3x^2 - 1)$
$= (2)(0.1) = 0.2$

43.) $f(x) = x^{-1}$, $x_0 = 0.5$, $dx = 0.1$

a.) $\Delta f = f(0.6) - f(0.5)$
$= (0.6)^{-1} - (0.5)^{-1} = \frac{-1}{3} \approx -0.3333$

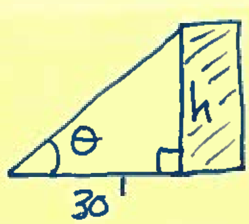
b.) $df = f'(0.5)\,dx \quad (\xrightarrow{D} f'(x) = -x^{-2} = \frac{-1}{x^2})$
$= \frac{-1}{(\frac{1}{2})^2} \cdot (0.1) = (-4)(0.1) = -0.4$

51.) $A = \pi r^2$, $r: 2 \to 2.02$ m. so
$\Delta r = 0.02$, $A' = 2\pi r$,
a.) $\Delta A \approx dA = A'(2) \cdot \Delta r = 4\pi(0.02)$
$= 0.08\pi$ m.^{2}
b.) % err. $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{0.08\cancel{\pi}}{\cancel{\pi}(2)^2} = \dfrac{0.08}{4}$
$= 0.02 = 2\%$

52.) a.) $C = \pi d \to$ diameter $d = \frac{1}{\pi} C$,
$d = 10$ in. $\to C = 10\pi$ in. ; $C: 10\pi \to 10\pi + 2$,
so $\Delta C = 2$ in ; $d' = \frac{1}{\pi}$, then
$\Delta d \approx d(d) = d'(10\pi) \cdot \Delta C = \frac{1}{\pi} \cdot 2 = \frac{2}{\pi}$ in.
b.) area $A = \pi r^2 = \pi\left(\frac{1}{2}d\right)^2 = \pi\left(\frac{1}{2}\left(\frac{1}{\pi}C\right)\right)^2 \to$
$A = \frac{1}{4\pi} C^2$ and $C: 10\pi \to 10\pi + 2$ so
$\Delta C = 2$ in. ; $A' = \frac{1}{2\pi} C$ then
$\Delta A \approx dA = A'(10\pi) \cdot \Delta C = (5) \cdot 2 = 10$ in.^{2}

53.) $V = \pi r^2 h = \pi r^2 (30) = 30\pi r^2$,
$r: 5.5 \to 6$ in. so $\Delta r = 0.5$;
$V' = 60\pi r$, then
$\Delta V \approx dV = V'(5.5) \cdot \Delta r = 330\pi\left(\frac{1}{2}\right) = 165\pi$ in.^{3}

54.)



$\tan\theta = \dfrac{h}{30} \to h = 30\tan\theta$

θ-values: $\theta \longrightarrow \theta + \Delta\theta$ (θ in radians!);
what must $|\Delta\theta|$ be in order that $\frac{|\Delta h|}{h} \le 4\%$? Then

$$\frac{|\Delta h|}{h} \approx \frac{|dh|}{h} = \frac{|h'(\theta)\cdot\Delta\theta|}{h} = \frac{|\cancel{30}\sec^2\theta\cdot\Delta\theta|}{\cancel{30}\tan\theta}$$

$$= \frac{\cancel{\cos\theta}}{\sin\theta}\cdot\frac{1}{\cos^{\cancel{2}}\theta}\cdot|\Delta\theta| \qquad (\theta = 75°)$$

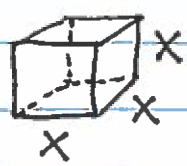
$$= \frac{1}{\sin 75°\cdot\cos 75°}\cdot|\Delta\theta| \le 4\% \longrightarrow$$

$$|\Delta\theta| \le 4\%(\sin 75°\cos 75°) = 0.04\left(\tfrac{1}{4}\right) = 0.01$$

$\longrightarrow \boxed{|\Delta\theta| \le 0.01 \text{ radians}}$ ($75° \approx 1.309$ rad.)

% error: $\frac{|\Delta\theta|}{\theta} \le \frac{0.01}{1.309} \approx 0.76\%$

56.)



Given $\frac{|dx|}{x} \le 0.5\%$

a.) $S = 6x^2 \xrightarrow{D} S' = 12x$; find $\frac{|\Delta S|}{S}$:

$$\frac{|\Delta S|}{S} \approx \frac{|dS|}{S} = \frac{|S'\cdot dx|}{S} = \frac{|12x\cdot dx|}{6x^2}$$

$$= 2\cdot\frac{|dx|}{x} \le 2(0.5\%) = 1\%$$

b.) $V = x^3 \xrightarrow{D} V' = 3x^2$; find $\frac{|\Delta V|}{V}$:

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V'\cdot dx|}{V} = \frac{|3x^2\cdot dx|}{x^3} = 3\cdot\frac{|dx|}{x}$$

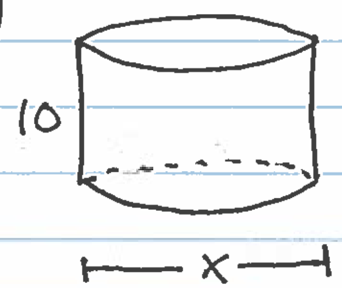
$$\le 3(0.5\%) = 1.5\%$$

57.) $V = \pi h^3$, h-values: $h \to h + \Delta h$,
$V' = 3\pi h^2$; assume $\frac{|\Delta V|}{V} \le 1\%$,
find $\frac{|\Delta h|}{h}$:

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V'(h) \cdot \Delta h|}{V} = \frac{3\cancel{\pi} h^2 \cdot |\Delta h|}{\cancel{\pi} h^3}$$

$$= 3 \cdot \frac{|\Delta h|}{h} \le 1\% \to \frac{|\Delta h|}{h} \le \frac{1}{3} \text{ of } 1\%$$

58.)

a.) 

$r = \frac{1}{2}x$

$V = \pi r^2 h$ (volume)
$= \pi \left(\frac{1}{2}x\right)^2 (10)$
$= \frac{5}{2}\pi x^2$; given
$\frac{|\Delta V|}{V} \le 1\%$; find $\frac{|dx|}{x}$:

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot dx|}{V}$$

$$= \frac{|\cancel{\tfrac{5}{2}\pi} \cdot 2x \cdot dx|}{\cancel{\tfrac{5}{2}\pi}\, x^2} = 2 \cdot \frac{|dx|}{x} \le 1\% \to$$

$$\frac{|dx|}{x} \le 0.5\%$$

b.) $S = 2\pi r h$ (surface area)
$= 2\pi \cdot \left(\frac{1}{2}x\right)(10) = 10\pi x$; given
$\frac{|\Delta S|}{S} \le 5\%$; find $\frac{|dx|}{x}$.

$$\frac{|\Delta S|}{S} \approx \frac{|dS|}{S} = \frac{|S' \cdot dx|}{S} = \frac{|\cancel{10\pi} \cdot dx|}{\cancel{10\pi}\, x}$$

$$= \frac{|dx|}{x} \le 5\%$$

59.)

$r = \frac{1}{2}x$

Given $\frac{|dx|}{x} = \frac{|\pm 1|}{100} = 1\%$;

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi\left(\frac{1}{2}x\right)^3 = \frac{1}{6}\pi x^3;$$

find $\frac{|\Delta V|}{V}$: $\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot dx|}{V}$

$$= \frac{|\cancel{\frac{1}{6}\pi} \cdot 3x^2 \cdot dx|}{\cancel{\frac{1}{6}\pi}\, x^3} = 3\frac{|dx|}{x} \le 3(1\%) = 3\%$$

62.) $C(t) = 1 + \frac{4t}{1+t^3} - e^{-0.06t} \xrightarrow{D}$

$$C'(t) = \frac{(1+t^3)(4) - (4t)(3t^2)}{(1+t^3)^2} + 0.06e^{-0.06t};$$

a.) $t: 20 \to 30$ so $dt = 30 - 20 = 10$;

find dC (ESTIMATE in change)

$dC = C'(20)\, dt$

$$= \left(\frac{-63{,}996}{64{,}016{,}001} + 0.06e^{-1.2}\right)(10)$$

$\approx (0.01707)(10)$

$= 0.1707$

b.) (EXACT change)

$C(30) - C(20)$

$= \left(1 + \frac{120}{27{,}001} - e^{-1.8}\right)$

$\quad - \left(1 + \frac{80}{8001} - e^{-1.2}\right)$

$\approx 0.8391 - 0.7088$

$= 0.1303$

67.) a.) $f(x) = 2^x \xrightarrow{D} f'(x) = 2^x \ln 2$,

so linearization at $x=0$ is

$$L(x) = f(0) + f'(0)(x-0)$$
$$= 2^0 + 2^0 \ln 2 \cdot x \quad = 1 + (\ln 2)x$$

68.) a.) $f(x) = \log_3 x \xrightarrow{D} f'(x) = \frac{1}{x} \cdot \frac{1}{\ln 3}$,

so linearization at $x=3$ is

$$L(x) = f(3) + f'(3)(x-3)$$
$$= \log_3 3 + \frac{1}{3} \cdot \frac{1}{\ln 3}(x-3)$$
$$= 1 + \frac{1}{3\ln 3} x - \frac{1}{\ln 3}$$
$$= 1 - \frac{1}{\ln 3} + \frac{1}{3\ln 3} x \quad .$$

1.) Estimate $\sqrt{90}$: Let $f(x) = \sqrt{x}$,

$x: 100 \longrightarrow 90$ so $\Delta x = -10$, $f'(x) = \frac{1}{2\sqrt{x}}$;

$\Delta f = f(90) - f(100) = \sqrt{90} - 10$,

$df = f'(100) \cdot \Delta x = \frac{1}{20}(-10) = -\frac{1}{2}$; assume

$\Delta f \approx df \longrightarrow \sqrt{90} - 10 \approx -\frac{1}{2} \longrightarrow$

$\boxed{\sqrt{90} \approx 9.5}$; calc : $\sqrt{90} \approx 9.487$ so

$\%$ err. $= \frac{9.5 - 9.487}{9.487} \approx 0.0014 = 0.14\%$.

2.) Estimate 2.0002^{25} : Let $f(x) = x^{25}$,

$x: 2 \to 2.0002$ so $\Delta x = 0.0002$,

$f'(x) = 25x^{24}$; then

$\Delta f = f(2.0002) - f(2) = 2.0002^{25} - 33{,}554{,}432$;

$df = f'(2) \cdot \Delta x = (419{,}430{,}400) \cdot (0.0002) = 83{,}886$;

assume $\Delta f \approx df \to$

$2.0002^{25} - 33{,}554{,}432 \approx 83{,}886 \to$

$\boxed{2.0002^{25} \approx 33{,}638{,}318}$;

calc: $2.0002^{25} \approx 33{,}638{,}418$ so

$$\% \text{ err.} = \frac{33{,}638{,}418 - 33{,}638{,}318}{33{,}638{,}418}$$

$\approx 0.000003 = 0.0003\ \%$

3.) Estimate $\cos(0.01)$: Let $f(x) = \cos x$,

$x: 0 \to 0.01$ so $\Delta x = 0.01$, $f'(x) = -\sin x$;

$\Delta f = f(0.01) - f(0) = \cos(0.01) - 1$,

$df = f'(0) \cdot \Delta x = (0) \cdot (0.01) = 0$; assume

$\Delta f \approx df \to \cos(0.01) - 1 \approx 0 \to$

$\boxed{\cos(0.01) \approx 1}$;

calc: $\cos(0.01) \approx 0.99995$ so

$$\% \text{ err.} = \frac{1 - 0.99995}{0.99995} \approx 0.00005 = 0.005\ \%$$

4.) Estimate $\ln(1.1)$:

Let $f(x) = \ln x$, $x : 1 \to 1.1$ so $\Delta x = 0.1$,

$f'(x) = \frac{1}{x}$;

$\Delta f = f(1.1) - f(1) = \ln(1.1) - \cancel{\ln 1}^{\,0}$,

$df = f'(1) \cdot \Delta x$

$= \frac{1}{(1)}(0.1) = 0.1$, assume $\Delta f \approx df$

$\to \ln 1.1 \approx 0.1$