Section 4.2

f is cont. on [0,1], diff. on (0,1)

1.) $f(x) = x^2 + 2x - 1$ on $[0,1]$; $\xrightarrow{D}$

$f'(x) = 2x + 2$, then

$f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{2 - (-1)}{1} = 3 \longrightarrow$

$2c + 2 = 3 \rightarrow 2c = 1 \longrightarrow c = \frac{1}{2}$

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f is cont. on [0,1], diff. on (0,1)

2.) $f(x) = x^{2/3}$ on $[0,1]$; $\xrightarrow{D}$ $f'(x) = \frac{2}{3} x^{-1/3}$,

then

$f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{1 - 0}{1} = 1 \longrightarrow$

$\frac{2}{3} c^{-1/3} = 1 \longrightarrow \dfrac{2}{c^{1/3}} = 3 \longrightarrow c^{1/3} = \frac{2}{3} \longrightarrow$

$c = \left(\frac{2}{3}\right)^3 \longrightarrow c = \frac{8}{27}$

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f is cont. on [-1,1], diff. on (-1,1)

5.) $f(x) = \arcsin x$ on $[-1,1]$; $\xrightarrow{D}$

$f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$, then

$f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}{2} = \dfrac{\pi}{2} \longrightarrow$

$\dfrac{1}{\sqrt{1 - c^2}} = \dfrac{\pi}{2} \longrightarrow \dfrac{2}{\pi} = \sqrt{1 - c^2} \longrightarrow \dfrac{4}{\pi^2} = 1 - c^2$

$\longrightarrow c^2 = 1 - \dfrac{4}{\pi^2} = \dfrac{\pi^2 - 4}{\pi^2} \longrightarrow c = \pm \dfrac{\sqrt{\pi^2 - 4}}{\pi}$

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f is cont. on [2,4], diff. on (2,4)

6.) $f(x) = \ln(x-1)$ on $[2,4]$; $\xrightarrow{D}$

$f'(x) = \dfrac{1}{x - 1}$, then $f'(c) = \dfrac{f(4) - f(2)}{4 - 2} \longrightarrow$

$\dfrac{1}{c - 1} = \dfrac{\ln 3 - \overset{0}{\cancel{\ln 1}}}{2} \longrightarrow 2 = \ln 3 \cdot c - \ln 3 \longrightarrow$

$$2 + \ln 3 = \ln 3 \cdot c \longrightarrow c = \frac{2 + \ln 3}{\ln 3}$$

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f is cont. on $[-1,2]$, diff. on $(-1,2)$

7.) $f(x) = x^3 - x^2$ on $[-1, 2]$; $\xrightarrow{D}$

$f'(x) = 3x^2 - 2x$, then

$$f'(c) = \frac{f(2) - f(-1)}{2-(-1)} = \frac{4-(-2)}{3} = 2 \longrightarrow$$

$$3c^2 - 2c = 2 \longrightarrow 3c^2 - 2c - 2 = 0 \longrightarrow$$

$$c = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

$$\approx 1.215 \text{ or } -0.549$$

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g is cont. on $[-2,2]$, diff. on $(-2,2)$

8.) $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$;

$\lim_{x \to 0^-} x^3 = 0^3 = 0 = g(0)$; $\lim_{x \to 0^+} x^2 = 0^2 = 0$, so

g cont. at $x = 0$; and $Dx^3 = 3x^2 = 0$ if $x = 0$, $Dx^2 = 2x = 0$ if $x = 0$, so g is diff. at $x = 0$; then

$$g'(c) = \frac{g(2) - g(-2)}{2-(-2)} = \frac{4-(-8)}{4} = 3 \longrightarrow$$

(why?)

$3c^2 = 3 \rightarrow c^2 = 1 \rightarrow \cancel{c=1}$ or $\boxed{c=-1}$

OR $2c = 3 \rightarrow \boxed{c = 3/2}$

9.) $f(x) = x^{2/3}$ on $[-1, 8]$; let $g(x) = x^2$ and $h(x) = x^{1/3}$ both of which are continuous for all values of x; and

$f(x) = x^{2/3} = (x^2)^{1/3} = h(x^2) = h(g(x))$

so f is continuous for all values of x (functional composition of continuous functions); BUT

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$ so f is NOT differentiable at $x = 0$; thus f is cont. for x in $[-1, 8]$, but is NOT differentiable for x in $(-1, 8)$ so the hypotheses of the MVT are <u>not</u> satisfied.

11.) $f(x) = \sqrt{x - x^2}$ on $[0, 1]$; let $g(x) = x - x^2$ and $h(x) = \sqrt{x}$ both of which are continuous for x in $[0, 1]$; and

$f(x) = \sqrt{x - x^2} = h(x - x^2) = h(g(x))$ is cont. for x in $[0, 1]$ (functional composition of continuous functions);

and $f'(x) = \frac{1}{2}(x - x^2)^{-1/2} \cdot (1 - 2x) = \frac{1 - 2x}{2\sqrt{x - x^2}}$

so f is differentiable on $(0,1)$;
then

$$f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{0-0}{1} = 0 \longrightarrow$$

$$\frac{1-2c}{2\sqrt{c-c^2}} = 0 \longrightarrow 1-2c=0 \longrightarrow c = 1/2 .$$

13.)

$$f(x) = \begin{cases} x^2 - x & , -2 \le x \le -1 \\ 2x^2 - 3x - 3 & , -1 < x \le 0 \end{cases}$$

$\lim_{x \to -1^+} (2x^2 - 3x - 3) = 2 + 3 - 3 = 2$,

$\lim_{x \to -1^-} (x^2 - x) = 1 - (-1) = 2 = f(-1)$, so f is cont. at $x = -1$;

$D(x^2 - x) = 2x - 1 = -3$ if $x = -1$,

$D(2x^2 - 3x - 3) = 4x - 3 = -7$ if $x = -1$;

so f is NOT diff. at $x=-1$; so f is cont. on $[-2,0]$, but f is not diff. on $(-2,0)$, so the hypotheses of the MVT are <u>not</u> satisfied.

16.)
$$f(x) = \begin{cases} 3 & , \text{ if } x = 0 \\ -x^2+3x+a & , \text{ if } 0 < x < 1 \\ mx+b & , \text{ if } 1 \le x \le 2 \end{cases} ;$$

<u>make f cont. at $x=0$</u> :

$\lim_{x \to 0^+} (-x^2+3x+a) = 3 \rightarrow \boxed{a=3}$;

<u>make f cont. at $x=1$</u> :

$\lim_{x \to 1^+} (mx+b) = \lim_{x \to 1^-} (-x^2+3x+a) \rightarrow$

$m+b = -1+3+3 \rightarrow \boxed{b = 5-m}$;

<u>make f diff. at $x=1$</u> :

$Y = -x^2+3x+a \xrightarrow{D} Y' = -2x+3 \rightarrow Y'(1) = 1$;

$Y = mx+b \xrightarrow{D} Y' = m$ so $\boxed{m=1} \rightarrow$

$\boxed{b=4}$; thus,

$$f(x) = \begin{cases} -x^2+3x+3 & , \text{ if } 0 \le x < 1 \\ x+4 & , \text{ if } 1 \le x \le 2 . \end{cases}$$

<u>Find values of c</u> :

$$f'(x) = \begin{cases} -2x+3, & \text{if } 0 \le x < 1 \\ 1, & \text{if } 1 \le x \le 2 \end{cases};$$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{6-3}{2} = \frac{3}{2} \rightarrow$$

<u>case 1</u>: $-2c+3 = \frac{3}{2} \rightarrow -2c = -\frac{3}{2} \rightarrow$

$c = \frac{3}{4}$

<u>case 2</u>: $1 = \frac{3}{2}$ (impossible)

23.) $g(t) = \sqrt{t} + \sqrt{1+t} - 4$, $0 < t < \infty$;

g is cont. (sum and composition of cont. fcns.) for $0 < t < \infty$;

$g(1) = \sqrt{1} + \sqrt{2} - 4 = \sqrt{2} - 3 < 0$ and

$g(16) = \sqrt{16} + \sqrt{17} - 4 = \sqrt{17} > 0$, so by IMVT there is <u>at least</u> <u>one</u> value c , $1 \le c \le 16$, with $g(c) = 0$;

Show there is <u>exactly</u> <u>one</u> value c satisfying $g(c) = 0$:

<u>METHOD I</u> : $\xrightarrow{D} g'(t) = \frac{1}{2}t^{-1/2} + \frac{1}{2}(1+t)^{-1/2}$

$= \dfrac{1}{2\sqrt{t}} + \dfrac{1}{2\sqrt{1+t}} > 0$ for $0 < t < \infty$,

so g is 1-1 $\rightarrow$ so the graph of g passes the horizontal line test $\rightarrow$ there is exactly one value c satisfying $g(c) = 0$.

<u>METHOD II</u> : Solve $\sqrt{t} + \sqrt{1+t} - 4 = 0$ for t $\rightarrow$ $4 - \sqrt{t} = \sqrt{1+t}$ $\rightarrow$ (square)

$16 - 8\sqrt{t} + \cancel{t} = 1 + \cancel{t}$ $\rightarrow$ $8\sqrt{t} = 15$ $\rightarrow$

$\sqrt{t} = \frac{15}{8}$ $\rightarrow$ $t = \frac{225}{64}$ $\rightarrow$ so exactly one solution.

METHOD III : (Proof by Contradiction)

assume there is a SECOND solution d with $g(d) = 0$, say $c < d$. apply the MVT to g on interval $[c, d]$. Then there is a # e, $c < e < d$, so that

$$g'(e) = \frac{g(d) - g(c)}{d - c} = \frac{0 - 0}{d - c} = 0 \ !$$

This is a contradiction since $g'(t) > 0$ for $0 < t < \infty$. Thus, there is exactly one solution.

26.) $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}$, $-\infty < \theta < \infty$;
r is cont. (prod. and sum of cont. fcns.) for $-\infty < \theta < \infty$;

$r(0) = 0 - \cos^2 0 + \sqrt{2} = -1 + \sqrt{2} > 0$,

$r(-\frac{\pi}{2}) = 2(-\frac{\pi}{2}) - \cos^2(-\frac{\pi}{2}) + \sqrt{2}$

$= -\pi - 0 + \sqrt{2} < 0$, so by IMVT there is <u>at least one</u> value c, $-\infty < c < \infty$, so that $r(c) = 0$; but

$r'(\theta) = 2 - 2\cos\theta \cdot (-\sin\theta)$

$= 2(1 + \sin\theta\cos\theta)$

$= 2(1 + \frac{1}{2}\sin 2\theta) > 0$

(since $-1 \le \sin 2\theta \le +1$) for $-\infty < c < \infty$, so r is 1-1. Thus there is <u>exactly one value</u> c satisfying $r(c) = 0$, since the graph of r passes the horizontal line test.

33.) a.) $y' = x$ so $y = \frac{1}{2}x^2 + c$

b.) $y' = x^2$ so $y = \frac{1}{3}x^3 + c$

c.) $y' = x^3$ so $y = \frac{1}{4}x^4 + c$

34.) a.) $y' = 2x$ so $y = x^2 + c$

b.) $y' = 2x - 1$ so $y = x^2 - x + c$

c.) $y' = 3x^2 + 2x - 1$ so $y = x^3 + x^2 - x + c$

35.) a.) $y' = \frac{-1}{x^2} = -x^{-2}$ so $y = x^{-1} + c = \frac{1}{x} + c$

b.) $y' = 1 - \frac{1}{x^2}$ so $y = x + \frac{1}{x} + c$

c.) $y' = 5 + \frac{1}{x^2}$ so $y = 5x - \frac{1}{x} + c$

36.) a.) $y' = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$ so $y = x^{1/2} + c$

b.) $y' = \frac{1}{\sqrt{x}}$ so $y = 2x^{1/2} + c$

c.) $y' = 4x - \frac{1}{\sqrt{x}}$ so $y = 2x^2 - 2x^{1/2} + c$

37.) a.) $y' = \sin 2t$ so $y = \frac{-1}{2}\cos 2t + c$

b.) $y' = \cos \frac{t}{2}$ so $y = 2 \sin \frac{t}{2} + c$

c.) $y' = \sin 2t + \cos \frac{t}{2}$ so

$y = \frac{-1}{2}\cos 2t + 2 \sin \frac{t}{2} + c$

38.) a.) $y' = \sec^2\theta$ so $y = \tan\theta + C$

b.) $y' = \theta^{1/2}$ so $y = \frac{2}{3}\theta^{3/2} + C$

c.) $y' = \sqrt{\theta} - \sec^2\theta$ so

$$y = \frac{2}{3}\theta^{3/2} - \tan\theta + C$$

39.) $f'(x) = 2x - 1 \rightarrow f(x) = x^2 - x + c$
and $x = 0, Y = 0 \rightarrow 0 = 0 - 0 + c \rightarrow$
$c = 0 \longrightarrow f(x) = x^2 - x$

40.) $g'(x) = \frac{1}{x^2} + 2x \longrightarrow g(x) = \frac{-1}{x} + x^2 + c$
and $x = -1, Y = 1 \longrightarrow 1 = \frac{-1}{-1} + 1 + c \rightarrow$
$c = -1 \longrightarrow g(x) = \frac{-1}{x} + x^2 - 1$

41.) $f'(x) = e^{2x} \longrightarrow f(x) = \frac{1}{2}e^{2x} + c$ and
$x = 0, Y = 3/2 \longrightarrow \frac{3}{2} = \frac{1}{2}(1) + c \rightarrow c = 1 \rightarrow$
$f(x) = \frac{1}{2}e^{2x} + 1$

42.) $r'(t) = \sec t \tan t - 1 \longrightarrow$
$r(t) = \sec t - t + c$ and $t = 0, r = 0$
$\longrightarrow 0 = \cancel{\sec} \overset{1}{0} - 0 + c \rightarrow c = -1 \longrightarrow$
$r(t) = \sec t - t - 1$

44.) $v = 32t - 2 \longrightarrow$
$S = 32 \cdot \frac{1}{2} t^2 - 2t + c$ and
$t = \frac{1}{2}, S = 4 \longrightarrow 4 = 16\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + c$
$\longrightarrow 4 = 4 - 1 + c \rightarrow c = 1 \longrightarrow$
$S = 16t^2 - 2t + 1$

46.) $V = \frac{2}{\pi} \cos \frac{2}{\pi} t \longrightarrow$

$s = \cancel{\frac{2}{\pi}} \cdot \frac{1}{\cancel{\frac{2}{\pi}}} \sin \frac{2}{\pi} t + c$ and

$t = \pi^2, \quad s = 1 \longrightarrow$

$1 = \sin\left(\frac{2}{\pi} \pi^2\right) + c \longrightarrow$

$1 = \underbrace{\sin 2\pi}_{0} + c \rightarrow c = 1 \longrightarrow$

$s = \sin \frac{2}{\pi} t + 1$

47.) $a(t) = e^t$ so $v(t) = e^t + c$ and

$v(0) = 20 \rightarrow e^0 + c = 20 \rightarrow c = 19 \rightarrow$

$v(t) = e^t + 19$ so $s(t) = e^t + 19t + c$

and $s(0) = 5 \rightarrow e^0 + 19(0) + c = 5 \rightarrow c = 4$

$\rightarrow \quad s(t) = e^t + 19t + 4$

48.) $a(t) = 9.8$ so $v(t) = 9.8t + c$ and

$v(0) = -3 \rightarrow 9.8(0) + c = -3 \rightarrow c = -3 \rightarrow$

$v(t) = 9.8t - 3$ so $s(t) = 4.9t^2 - 3t + c$

and $s(0) = 0 \rightarrow 4.9(0)^2 - 3(0) + c = 0 \rightarrow$

$c = 0 \rightarrow \quad s(t) = 4.9t^2 - 3t$

51.) Let $T(t)$ be the temperature (°C) at time t seconds; then by MVT

$$T'(c) = \frac{T(14) - T(0)}{14 - 0} = \frac{(100) - (-19)}{14} = 8.5 \frac{°C}{sec.}$$

so at time c, $0 \le c \le 14$, the temperature is increasing at the rate of 8.5 °C/sec.

52.) Let $L(t)$ be distance (mi.) traveled after t hours; then by MVT

$$L'(c) = \frac{L(2) - L(0)}{2 - 0} = \frac{159 - 0}{2} = 79.5 \text{ mph}$$

so at time c, $0 \le c \le 2$, speed of truck $L'(c) = 79.5$ mph.

54.) Let $s = s(t)$ be runner's distance (mi.) after t hours, then

$$ARC = \frac{s(2.2) - s(0)}{2.2 - 0} = \frac{26.2}{2.2} \approx 11.91 \text{ mph};$$

so by MVT there is some time c so that $s'(c) = 11.91$ mph (IRC); and $s'(0) = 0$ mph and $s'(2.2) = 0$ mph. If we assume s' is continuous then by IMVT ($m = 11$ is between 0 and 11.91), then TWICE $s' = 11$ mph!

56.) $s''(t) = -1.6 \text{ m/sec.}^2$, $s'(0) = 0$ m./sec. (dropped), then

$s'(t) = -1.6t + c$ ($t = 0$, $s' = 0$

$\rightarrow 0 = -1.6(0) + c \rightarrow c = 0$) $\rightarrow$

$s'(t) = -1.6t$;

$s'(30) = -1.6(30) = -48$ m./sec.

(speed is 48 m./sec.)

57.) apply MVT to $f(x) = \frac{1}{x}$ on $[a,b]$: $\xrightarrow{D} f'(x) = \frac{-1}{x^2}$,

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a}$$

$$= \frac{a-b}{ab}\cdot\frac{1}{b-a} = \frac{-1}{ab}, \text{ i.e.,}$$

$$\frac{-1}{c^2} = \frac{-1}{ab} \rightarrow c^2 = ab \rightarrow c = \sqrt{ab}$$

(since a and b are positive)

58.) apply MVT ~~to~~ $f(x) = x^2$ on $[a,b]$: $\xrightarrow{D}$ $f'(x) = 2x$,

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b^2-a^2}{b-a}$$

$$= \frac{\cancel{(b-a)}(b+a)}{\cancel{b-a}}, \text{ i.e., } 2c = a+b \rightarrow$$

$$c = \frac{1}{2}(a+b)$$

63.) assume f cont. on $[1,4]$ and diff. on $(1,4)$, then by MVT

$$f'(c) = \frac{f(4)-f(1)}{4-1} = \frac{f(4)-f(1)}{3} \leq 1$$

$$\rightarrow f(4)-f(1) \leq 3$$

65.) assume $f(t) = \cos t$ is cont. on $[0,x]$ and diff. on $(0,x)$, then

$f'(t) = -\sin t$ and

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\cos x - \cos 0}{x} \rightarrow$$

$$-\sin c = \frac{\cos x - 1}{x} \rightarrow$$

$$\frac{|\cos x - 1|}{|x|} = |-\sin c| \leq 1 \rightarrow$$

$$|\cos x - 1| \leq |x| .$$

66.) Let $f(x) = \sin x$ on $[a, b]$; by MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a} \rightarrow$$

$$\cos(c) = \frac{\sin b - \sin a}{b - a} \rightarrow$$

$$\frac{|\sin b - \sin a|}{|b - a|} = |\cos c| \leq 1 \rightarrow$$

$$|\sin b - \sin a| \leq |b - a| .$$

73.) assume f' is defined for all x, $f(1) = 1$, $f' < 0$ for $x < 1$, and $f' > 0$ for $x > 1$.

a.) Show $f(x) \geq 1$ for all x:
Let $x > 1$ and apply MVT to f on $[1, x] \rightarrow$

$$f'(c) = \frac{f(x) - f(1)}{x - 1}$$

$$= \frac{f(x) - 1}{x - 1} > 0 \rightarrow$$

$f(x) - 1 > 0 \rightarrow f(x) > 1$ for $x > 1$.

Let $x < 1$ and apply MVT to f on $[x, 1] \rightarrow$

$$f'(c) = \frac{f(1) - f(x)}{1 - x}$$

$$= \frac{1 - f(x)}{1 - x} < 0 \rightarrow$$

$1 - f(x) < 0 \rightarrow f(x) > 1$ for $x < 1$.
Since $f(1) = 1$, we conclude

$f(x) \geq 1$ for all x.

b.) Show $f'(1) = 0$:

$$f'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(1+h) - 1}{h} \begin{matrix} \leftarrow (+) \\ \leftarrow (+) \end{matrix} \geq 0,$$

and

$$f'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^-} \frac{f(1+h) - 1}{h} \begin{matrix} \leftarrow (+) \\ \leftarrow (-) \end{matrix} \leq 0,$$

so $0 \leq f'(1) \leq 0 \longrightarrow$

$$f'(1) = 0.$$