Section 2.5

2.) NOT continuous at $x = 3$ since 
\[ \lim_{{x \to 3}} f(x) = 1 \neq f(3) = 1.5 \]

4.) NOT continuous at $x = 1$ since 
\[ \lim_{{x \to 1^+}} f(x) = 0 \text{ and } \lim_{{x \to 1^-}} f(x) = 1.5 \]
so that \( \lim_{{x \to 1}} f(x) \) DOES NOT EXIST

5.) a.) YES  \quad f(-1) = 0
   b.) YES  \quad \lim_{{x \to -1^+}} f(x) = 0
   c.) YES
   d.) YES

6.) a.) YES  \quad f(1) = 1
   b.) YES  \quad \lim_{{x \to 1}} f(x) = 2
   c.) NO
   d.) NO

7.) a.) NO
   b.) NO

8.) $f$ continuous on interval $[0, 3]\]
   except $x = 0$, $x = 1$, $x = 2$, and $x = 3$

9.) Let $f(2) = 0$
   10.) Let $f(1) = 2$
16.) \( Y = \frac{x+3}{x^2-3x-10} \); \( Y = x+3 \) and \( Y = x^2-3x-10 \) are continuous for all values of \( x \) since they are polynomials; therefore, since \( Y = \frac{x+3}{x^2-3x-10} \) is the quotient of these functions, it is continuous for all values of \( x \) except where \( x^2-3x-10 = (x-5)(x+2) = 0 \), i.e., except for \( x = 5 \) and \( x = -2 \).

20.) \( Y = \frac{x+2}{\cos x} \); \( Y = x+2 \) is continuous for all values of \( x \) since it is a polynomial; \( Y = \cos x \) is continuous for all values of \( x \) since it is a well-known trig function; therefore, since \( Y = \frac{x+2}{\cos x} \) is the quotient of these functions, it is continuous for all values of \( x \) except where \( \cos x = 0 \), i.e., except for \( x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots \).

26.) \( Y = (3x-1)^{\frac{1}{4}} \); let \( f(x) = x^{\frac{1}{4}} \), which is continuous for \( x \geq 0 \), and let \( g(x) = 3x-1 \), which is continuous for all values of \( x \) since it is a polynomial;
since \( y = (3x - 1)^{1/4} = f(3x - 1) = f(g(x)) \)

is functional composition, it is continuous for all \( x \)-values for which \( 3x - 1 \geq 0 \), i.e., for \( x \geq \frac{1}{3} \)

29.) \( g(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & \text{if } x \neq 3 \\ 5 & \text{if } x = 3 \end{cases} \)

\( = \begin{cases} \frac{(x-3)(x+2)}{x-3} & \text{if } x \neq 3 \\ \frac{x+2}{5} & \text{if } x = 3 \end{cases} \)

\( y = x + 2 \) (line) is continuous for all \( x \)-values. Check \( x = 3 \):

i.) \( g(3) = 5 \)

ii.) \( \lim_{x \to 3} g(x) = \lim_{x \to 3} (x + 2) = 3 + 2 = 5 \)

iii.) \( \lim_{x \to 3} g(x) = 5 = g(3) \), so \( g \) is continuous at \( x = 3 \); thus \( g \) is continuous for all \( x \)-values

30.) \( f(x) = \begin{cases} \frac{x^3 - 8}{x^2 - 4} & \text{if } x \neq 2, x \neq -2 \\ 3 & \text{if } x = 2 \\ 4 & \text{if } x = -2 \end{cases} \)
\[
\begin{cases}
\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}, & x \neq 2, x \neq -2 \\
3 & x = 2 \\
4 & x = -2
\end{cases}
\]

\[
\begin{cases}
\frac{x^2+2x+4}{x+2}, & x \neq 2, x \neq -2 \\
3 & x = 2 \\
4 & x = -2
\end{cases}
\]

\(y = x^2 + 2x + 4\) (parabola) and 
\(y = x + 2\) (line) are continuous
for all \(x\)-values, so
\[
\frac{x^2+2x+4}{x+2}
\]
is continuous for all \(x\)-values except \(x = -2\).

**Check \(x = 2\):**

i.) \(f(2) = 3\)

ii.) 
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3
\]

iii.) \(\lim_{x \to 2} f(x) = 3 = f(2)\); thus,

\(f\) is continuous at \(x = 2\).

**Check \(x = -2\):**

i.) \(f(-2) = 4\)

ii.) 
\[
\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2+2x+4}{x+2} = \frac{4}{0} = \pm \infty
\]

so \(f\) is **not** continuous at \(x = -2\);

\(f\) is continuous at all \(x\)-values EXCEPT \(x = -2\).
42.) \( g(x) = \frac{x^2 - 16}{x^2 - 3x - 4} \) then

\[
\lim_{x \to 4} g(x) = \lim_{x \to 4} \frac{x^2 - 16}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)(x+1)} = \frac{8}{5},
\]

so define \( g(4) = \frac{8}{5} \) and \( g \) will be continuous at \( x = 4 \).

43.) Let \( f(x) = \begin{cases} x^2 - 1, & \text{if } x < 3 \\ 2ax, & \text{if } x \geq 3 \end{cases} \)

\[ Y = x^2 - 1 \text{ is continuous for } x < 3 \text{ (polynomial)}; \]

\[ Y = 2ax \text{ is continuous for } x > 3 \text{ (line)}; \]

make \( f \) continuous at \( x = 3 \) by forcing limits to be equal:

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 1) = 9 - 1 = 8,
\]

\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2ax) = 6a, \quad \text{so}
\]

\[ 6a = 8 \implies a = \frac{4}{3}. \]
44. Let \( g(x) = \begin{cases} x, & \text{if } x < -2 \\ 6x^2, & \text{if } x \geq -2 \end{cases} \)

\( y = x \) is continuous for \( x < -2 \) (line),
\( y = 6x^2 \) is continuous for \( x \geq -2 \) (parabola); make \( g \) continuous at \( x = -2 \) by forcing limits to be equal:

\[
\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x = -2,
\]

\[
\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} 6x^2 = 4b,
\]

so \( 4b = -2 \rightarrow b = -\frac{1}{2} \)

47. Let \( f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ ax - b, & \text{if } -1 < x < 1 \\ 3, & \text{if } x \geq 1 \end{cases} \)

We need

\[
\lim_{x \to 1^-} (ax - b) = 3 \quad \text{and} \quad \lim_{x \to -1^+} (ax - b) = -2 \rightarrow
\]

\[
\begin{cases}
a(-1) - b = 3 \\
a(1) - b = -2
\end{cases} \rightarrow \begin{cases}
a = b + 3 \\
a = b - 2
\end{cases} \rightarrow
\]

\[
0 = 2b + 1 \rightarrow b = -\frac{1}{2} \quad \text{and} \quad a = \frac{-1}{2} + 3 = \frac{-1}{2} + \frac{6}{2} \rightarrow a = \frac{5}{2}
\]
48.) Let \( g(x) = \begin{cases} 
x + 2b, & \text{if } x \leq 0 \\
x^2 + 3a - b, & \text{if } 0 < x \leq 2 \\
3x - 5, & \text{if } x > 2 
\end{cases} \)

We need
\[
\lim_{{x \to 0^+}} (ax + 2b) = \lim_{{x \to 0^+}} (x^2 + 3a - b) \\
\lim_{{x \to 2^-}} (x^2 + 3a - b) = \lim_{{x \to 2^-}} (3x - 5)
\]

\[
\begin{align*}
\Rightarrow & \quad a(0) + 2b = (0)^2 + 3a - b \\
\Rightarrow & \quad (2)^2 + 3a - b = 3(2) - 5 \\
\Rightarrow & \quad 2b = 3a - b \\
\Rightarrow & \quad 4 + 3a - b = 1 \\
\Rightarrow & \quad 3b = 3a \quad \Rightarrow \quad 3a - b + 3 = 0 \\
\Rightarrow & \quad a = b \quad (5\text{vi}) \quad \Rightarrow \quad 3(b) - b + 3 = 0 \\
\Rightarrow & \quad 2b + 3 = 0 \quad \Rightarrow \quad b = -\frac{3}{2} \quad , \quad a = -\frac{3}{2}
\end{align*}
\]

52.) **RECALL:** i. \( \lim_{{n \to \infty}} (1 + \frac{1}{n})^n = e \\
ii. \lim_{{x \to 0}} (1 + x)^{\frac{1}{x}} = e 
\)

For \( f(x) = (1 + 2x)^{\frac{1}{x}} \):
\[
g(x) = \ln (1 + 2x) \text{ is cont. for } x > -\frac{1}{2};
\]
\[ h(x) = \frac{\ln(1 + 2x)}{x} \text{ is cont. (quotient)} \]
for all \( x > -\frac{1}{2} \) \text{ EXCEPT at } x = 0 \; ;

\[ k(x) = e^x \text{ is (well-known) cont. } \]
for all \( x \)-values \; ; \text{ then}

\[ f(x) = (1 + 2x)^{\frac{1}{x}} = k(h(x)) \]

\[ = e^{h(x)} \]

\[ = e^{\frac{1}{x} \ln(1 + 2x)} \]

\[ = e^{\frac{\ln(1 + 2x)}{x}} = (1 + 2x)^{\frac{1}{x}} \]

is cont. (composition) for all \( x > -\frac{1}{2} \) \text{ EXCEPT at } x = 0 \; ; \text{ and}

\[ \lim_{x \to 0} (1 + 2x)^{\frac{1}{x}} = \lim_{x \to 0} \left[ (1 + 2x)^{\frac{1}{2x}} \right]^2 \]

\[ = e^2 \; ; \text{ so} \]

\[ \boxed{f(0) = e^2} \] \text{ and } \( f \) will \text{ be cont. at } x = 0 \; .

56.) \( F(x) = (x-a)^2 (x-b)^2 + x \) \; ;
\( F \) is cont. (polynomial) for all \( x \)-values \; ; assume \( a < b \) and consider the interval \( [a, b] \) \; .
\[ F(a) = (c)^2(a-b)^2 + a = a \quad \text{and} \]
\[ F(b) = (b-a)^2(c)^2 + b = b \quad \text{and} \]
\[ m = \frac{a+b}{2} \quad \text{is between} \]
\[ F(a) \quad \text{and} \quad F(b) \quad \text{thus by IMVT} \]
\[ \text{there is at least one } x \text{-value} \]
\[ c \quad \text{so that} \quad F(c) = m \quad \text{i.e.} \]
\[ F(c) = \frac{a+b}{2} \quad \text{and} \quad c \text{ is in } [a,b]. \]

59.) Let \( f(x) = \begin{cases} \frac{\sin(x-2)}{x-2}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases} \)
\[ y = \frac{\sin(x-2)}{x-2} \quad \text{is cont. (quotient)} \]
\[ \text{for all } x \text{-values EXCEPT at } x = 2 \; j \]
\[ \lim_{x \to 2} \frac{\sin(x-2)}{x-2} = 1, \quad \text{but } f(2) = 0, \]
\[ \text{so } f \text{ has a removable discontinuity at } x = 2. \]

62.) Let \( f(x) = x \) and \( g(x) = x - \frac{1}{2} \)
\[ \text{then both } f \text{ and } g \text{ are cont. for } 0 \leq x \leq 1, \quad \text{but} \]
\[ \frac{f(x)}{g(x)} = \frac{x}{x-\frac{1}{2}} \text{ is cont for } 0 \leq x \leq 1 \]
EXCEPT at \( x = \frac{1}{2} \) (YES)

63. Let \( f(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases} \) and

\[ g(x) = \begin{cases} 2, & \text{if } x < 0 \\ -2, & \text{if } x \geq 0 \end{cases} \]

then \( h(x) = f(x)g(x) \)

\[ = \begin{cases} (1)(2), & \text{if } x < 0 \\ (1)(-2), & \text{if } x \geq 0 \end{cases} = \begin{cases} 2, & \text{if } x < 0 \\ -2, & \text{if } x \geq 0 \end{cases} = 2, \]

which is cont. at \( x = 0 \), but neither \( f \) nor \( g \) is cont. at \( x = 0 \).

71. We have \( x^3 - 3x - 1 = 0 \), so let \( f(x) = x^3 - 3x - 1 \) and \( m = 0 \); \( f \) is cont. (polynomial) for all \( x \)-values; \( f(0) = -1 \) and \( f(2) = +1 \) and \( m = 0 \) is between \( f(0) \) and \( f(2) \) so choose interval \([0, 2] \); thus by IMVT there is at least one \( x \)-value \( c \), \( 0 \leq c \leq 2 \), so that \( f(c) = m \), i.e., \( c^3 - 3c - 1 = 0 \) and equation is solvable.

74. We have \( x^x = 2 \), so let \( f(x) = x^x \) and \( m = 2 \); \( g(x) = e^x \).
is cont. (well known) for all \( x \) values; \( h(x) = \ln x \) is cont. (well known) for \( x > 0 \);
\( k(x) = x \ln x \) is cont. (product) for \( x > 0 \); hence
\[
f(x) = x^x = g(k(x))
= e^{k(x)}
= e \times \ln x
= e^{\ln x^x} = x^x \text{ is cont. (composition) for } x > 0 \text{; } f(1) = 1^1 = 1
\text{ and } f(2) = 2^2 = 4 \text{ and } m = 2 \text{ is between } f(1) \text{ and } f(2) \text{ so choose interval } [1, 2]. \text{ Thus, by IMVT there is at least one } x \text{-value } c, 1 \leq c \leq 2, \text{ so that } f(c) = m, i.e., c^c = 2, \text{ and the equation is solvable.}

75.) \text{ We have } \sqrt{x} + \sqrt{1+x} = 4, \text{ so let } \ f(x) = \sqrt{x} + \sqrt{1+x} \text{ and } m = 4; \ g(x) = \sqrt{x} \text{ is cont. for } x \geq 0 \text{ (well known); } h(x) = \sqrt{1+x} \text{ is cont. (composition) for } x \geq -1 \text{; so } f \text{ is cont. (sum).}
For \( x > -1 \); \( f(0) = \sqrt{0} + \sqrt{1} = 1 \) and
\( f(16) = \sqrt{16} + \sqrt{17} = 4 + \sqrt{17} \) and \( m = 4 \)
is between \( f(0) \) and \( f(16) \) so we use interval \([0, 16]\); they,
by IVP there is at least one \( x \)-value \( c \), \( 0 \leq c \leq 16 \),
so that \( f(c) = m \), i.e.,
\( \sqrt{c} + \sqrt{1+c} = 4 \) and the equation is solvable.
I.) Prove $x^3 = x + 2$ is solvable:

$x^3 = x + 2 \Rightarrow x^3 - x - 2 = 0$, so let $f(x) = x^3 - x - 2$ and $m = 0$; note that $f(1) = -2 < 0$ and $f(2) = 4 > 0$ so $m = 0$ is between $f(1)$ and $f(2)$; use the interval $[1, 2]$; $f$ is a continuous function on $[1, 2]$ since it is a polynomial. By the IVT it follows that there is a number $c$, $1 \leq c \leq 2$, so that $f(c) = m$, i.e., $c^3 - c - 2 = 0$, and the original equation is solvable.
II.) Prove $2 + \sin x = x$ is solvable:

$2 + \sin x = x \Rightarrow 2 - x + \sin x = 0$ so let $f(x) = 2 - x + \sin x$ and $m = 0$; $f$ is continuous for all values of $x$ since it is the sum of continuous functions ($y = 2 - x$, a line, and $y = \sin x$, a well-known trig function); note that $f(0) = 2 > 0$ and $f(\pi) = 2 - \pi - \sin \pi = 2 - \pi < 0$, so $m = 0$ is between $f(0)$ and $f(\pi)$, use the interval $[0, \pi]$. By the IMVT it follows that there is a number $c$, $0 \leq c \leq \pi$, so that $f(c) = m$, i.e., $2 - c + \sin c = 0$, and the original equation is solvable.
1.) Use limits and algebra to determine the value of constants \( A \) and \( B \) so that each of the following functions is continuous for all values of \( x \).

a.) \( f(x) = \begin{cases} \frac{x^2 - 7x + 6}{x - 6}, & \text{if } x \neq 6 \\ A, & \text{if } x = 6. \end{cases} \)

b.) \( f(x) = \begin{cases} A^2 x - A, & \text{if } x \geq 1 \\ 2, & \text{if } x < 1. \end{cases} \)

c.) \( f(x) = \begin{cases} \frac{A + x}{A + 1}, & \text{if } x < 0 \\ A x^3 + 3, & \text{if } x \geq 0. \end{cases} \)

d.) \( f(x) = \begin{cases} 3, & \text{if } x \leq 1 \\ A x^2 + B, & \text{if } 1 < x \leq 2 \\ 5, & \text{if } x > 2. \end{cases} \)

e.) \( f(x) = \begin{cases} A x - B, & \text{if } x \leq -1 \\ 2 x + 3 A + B, & \text{if } -1 < x \leq 1 \\ 4, & \text{if } x > 1. \end{cases} \)

2.) For what \( x \)-values are the following functions continuous? Briefly explain why using shortcuts and rules from class. Sketch the graph of each using a graphing calculator.

a.) \( g(x) = \frac{x + 1}{x^2 - 4} \)

b.) \( h(x) = \frac{100}{4 - \sqrt{x^2 - 9}} \)

c.) \( h(x) = \sin^3(\ln(3x - 5)) \)

d.) \( g(x) = \begin{cases} \frac{x^2 - 3x - 4}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4. \end{cases} \)

e.) \( f(x) = \begin{cases} \frac{x^3 + 1}{x^2 - 1}, & \text{if } x \neq 1, -1 \\ -3/2, & \text{if } x = -1 \\ 3, & \text{if } x = 1. \end{cases} \)
1.) a.) Since \( \lim_{x \to 6} f(x) = \lim_{x \to 6} \frac{x^2 - 7x + 6}{x - 6} \)

   \[ \lim_{x \to 6} \frac{(x-6)(x-1)}{(x-6)} = 5 \]

   Thus, choosing \( a = 5 \)

   makes \( f \) continuous at \( x = 6 \) (it's already continuous for \( x \neq 6 \).)

b.) \( f \) is continuous for \( x < 1 \) and for \( x > 1 \).

   We must make \( f \) continuous at \( x = 1 \):

   \[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (a^2x - a) = a^2 - a \quad \text{and} \]

   \[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2) = 2 \]

   Thus, \( a^2 - a = 2 \)

   \[ a^2 - a - 2 = 0 \rightarrow (a - 2)(a + 1) = 0 \rightarrow a = 2 \quad \text{or} \quad a = -1 \]

c.) \( f \) is continuous for \( x < 0 \) (so long as \( a \neq -1 \))

   and for \( x > 0 \). We must make \( f \)

   continuous at \( x = 0 \):

   \[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (ax^3 + 3) = 3 \quad \text{and} \]

   \[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{a + x}{a + 1} = \frac{a}{a + 1} \quad \text{thus} \frac{a}{a + 1} = 3 \rightarrow \]

   \[ a = 3a + 3 \rightarrow -3 = 2a \rightarrow a = \frac{-3}{2} \]
d.) \( f \) is continuous for \( x < 1 \), for \( 1 < x < 2 \), and for \( x > 2 \). We must make \( f \) continuous at \( x = 1 \) and at \( x = 2 \):

at \( x = 1 \): \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} (ax^2 + b) = a + b \) and
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (3) = 3, \quad \text{and} \quad a + b = 3.
\]

at \( x = 2 \): \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (ax^2 + b) = 4a + b \) so
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} (ax^2 + b) = 5, \quad \text{and} \quad 4a + b = 5.
\]

Thus \( a + b = 3 \) \( \Rightarrow \) \( b = 3 - a \)
\[
4a + b = 5 \quad \Rightarrow \quad 4a + (3 - a) = 5 \quad \Rightarrow \quad 3a = 2 \quad \Rightarrow \quad a = \frac{2}{3} \quad \text{and} \quad b = \frac{7}{3}.
\]

e.) \( f \) is continuous for \( x < -1 \), for \( -1 < x < 1 \), and for \( x > 1 \). We must make \( f \) continuous at \( x = -1 \) and \( x = 1 \):

at \( x = -1 \): \( \lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} (2x + 3a + b) = 3a + b - 2 \) and
\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} (a - b) = -a - b \), so \( 3a + b - 2 = -a - b \)
\]

at \( x = 1 \): \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (4) = 4 \) and
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (2x + 3a + b) = 2 + 3a + b, \quad \text{so} \quad 2 + 3a + b = 4.
\]
Thus, \[3a + b - 2 = -a - b \quad \text{and} \quad 4a + 2b = 2\]
\[\rightarrow 2 + 3a + b = 4 \quad \text{and} \quad 3a + b = 2 \quad \therefore b = 2 - 3a\]
\[\rightarrow 4a + 2(2 - 3a) = 2 \quad \rightarrow 4a + 4 - 6a = 2 \quad \rightarrow 2 = 2a \quad \therefore a = 1 \quad \text{and} \quad b = -1\]

2.) a.) \(Y = x + 1\) and \(Y = x^2 - 4\) are continuous for all values of \(x\) (since they are polynomials), so \(g(x) = \frac{x + 1}{x^2 - 4}\) is continuous for all values of \(x\) (quotient of continuous functions) except where \(x^2 - 4 = (x - 2)(x + 2) = 0\), i.e., except for \(x = 2\) and \(x = -2\).

b.) \(Y = x^2 - 9\) and \(Y = 100\) are continuous for all values of \(x\) (since they are polynomials); \(Y = \sqrt{x}\) is a well
Known continuous function for \( x \geq 0 \); let \( f(x) = \sqrt{x} \) and \( g(x) = x^2 - 9 \), then \( \sqrt{x^2 - 9} = f(g(x)) \) is continuous (composition of continuous functions) so long as \( x^2 - 9 \geq 0 \), i.e., \( (x-3)(x+3) \geq 0 \), i.e., for \( x \geq 3 \) and \( x \leq -3 \);

\[
Y = Y \text{ is continuous for all values of } x, \text{ so that } Y = 4 - \sqrt{x^2 - 9} \text{ is continuous (difference of continuous functions)} \text{ for } x \geq 3 \text{ and } x \leq -3 \; \text{; finally},
\]

\[
h(x) = \frac{100}{4 - \sqrt{x^2 - 9}} \text{ is continuous (quotient of continuous functions)} \text{ for } x \geq 3 \text{ and } x \leq -3 \text{ so long as } 4 - \sqrt{x^2 - 9} \neq 0 \; \text{;}
\]

\[
4 - \sqrt{x^2 - 9} = 0 \Rightarrow 4 = \sqrt{x^2 - 9} \Rightarrow 16 = x^2 - 9 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5 \; \text{; thus},
\]

\[
h(x) = \frac{100}{4 - \sqrt{x^2 - 9}} \text{ is continuous for } x \geq 3 \text{ and } x \leq -3 \text{ except } x = \pm 5.
\]
c.) \( y = 3x - 5 \) and \( y = x^3 \) are continuous for all values of \( x \) (since they are polynomials), and \( y = \sin x \) is a well-known function continuous for all values of \( x \); \( y = \ln x \) is a well-known function continuous for \( x > 0 \).

\[ y = \ln x \]

Let \( f(x) = \ln x \) and \( g(x) = 3x - 5 \), then \( \ln (3x - 5) = f(g(x)) \) is continuous (composition of continuous functions) so long as \( 3x - 5 > 0 \), i.e., for \( x > \frac{5}{3} \).

Let \( k(x) = x^3 \) and \( l(x) = \sin x \), then \( h(x) = \sin^3 (\ln (3x - 5)) = k(l(f(g(x)))) \) is continuous (composition of continuous functions) for \( x > \frac{5}{3} \).

For graph of function try the following ranges:

1. \( \frac{5}{3} < x \leq 1000 \)
2. \( \frac{5}{3} < x \leq 100 \)
3. \( \frac{5}{3} < x \leq 10 \)
4. \( \frac{5}{3} < x \leq 2 \)
5. \( \frac{5}{3} < x \leq 1.75 \)
6. \( \frac{5}{3} < x \leq 1.68 \)
7. \( \frac{5}{3} < x \leq 1.668 \)
8. \( \frac{5}{3} < x \leq 1.666 \)
d.) \( g(x) = \begin{cases} \frac{x^2 - 3x - 4}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4 \end{cases} \)

\[= \begin{cases} \frac{(x - 4)(x + 1)}{x - 4}, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4 \end{cases}, \]

\[= \begin{cases} x + 1, & \text{if } x \neq 4 \\ 5, & \text{if } x = 4 \end{cases} \]

i.) \( g(4) = 5 \)

ii.) \( \lim_{x \to 4} g(x) = \lim_{x \to 4} (x + 1) = 4 + 1 = 5 \)

iii.) \( \lim_{x \to 4} g(x) = g(4) \)

Thus, \( g \) is continuous at \( x = 4 \); since \( y = x + 1 \) is continuous for \( x \neq 4 \) (since it is a polynomial), \( g \) is continuous for all values of \( x \).

\[
\begin{array}{c}
y = x + 1 \\
\end{array}
\]

\[
\begin{array}{c}
x \end{array}
\]

\[
\end{array}
\]

\[
\begin{array}{c}
y = \frac{x^3 + 1}{x^2 - 1}, & \text{if } x \neq 1, -1 \\
-\frac{3}{2}, & \text{if } x = -1 \\
3, & \text{if } x = 1
\end{cases}
\]
\[ y = x^3 + 1 \] and \[ y = x^2 - 1 \] are continuous for all values of \( x \) (since they are polynomials), so \[ y = \frac{x^3 + 1}{x^2 - 1} \] is continuous for all values of \( x \) except where \( x^2 - 1 = 0 \), i.e., except for \( x = \pm 1 \).

Check \( x = 1 \): i.) \( f(1) = 3 \), ii.) \[ \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^3 + 1}{x^2 - 1} = \frac{2}{0^+} = \pm \infty \] so \[ \lim_{x \to 1} f(x) \] does NOT exist and \( f \) is NOT cont. at \( x = 1 \).

Check \( x = -1 \): i.) \( f(-1) = \frac{-3}{2} \), ii.) \[ \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^3 + 1}{x^2 - 1} = \lim_{x \to -1} \frac{(x+1)(x^2 - x + 1)}{(x+1)(x-1)} = \frac{3}{-2} = -\frac{3}{2} \] and iii.) \( f(-1) = \lim_{x \to -1} f(x) \) so that \( f \) is continuous at \( x = -1 \); thus, \( f \) is continuous for all \( x \)-values except \( x = 1 \).
\[ \lim_{x \to 1^+} f(x) = \frac{2}{0^+} = +\infty, \]
\[ \lim_{x \to 1^-} f(x) = \frac{2}{0^-} = -\infty, \]
\[ \lim_{x \to +\infty} f(x) = +\infty, \]
\[ \lim_{x \to -\infty} f(x) = -\infty. \]