Math 21A
Kouba
A Useful Limit

Problem: Determine \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \), where \( \theta \) is given in radians.

Start by assuming that \( \theta > 0 \) is in the first quadrant, and assume the circle has radius 1.

It follows that

\[
\text{Area } \triangle OAD < \text{Area } \triangle OAC < \text{Area } \triangle OBC \rightarrow
\]

\[
\frac{1}{2}(\cos \theta)(\sin \theta) < \frac{\theta}{\frac{\pi}{2}} \cdot \pi(1)^2 < \frac{1}{2}(1)(\tan \theta) \rightarrow
\]

\[
(\cos \theta)(\sin \theta) < \theta < \frac{\sin \theta}{\cos \theta} \rightarrow
\]

\[
\cos \theta < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \rightarrow
\]

\[
\frac{1}{\cos \theta} > \frac{\sin \theta}{\theta} > \cos \theta \rightarrow
\]

\[
\lim_{\theta \to 0} \frac{1}{\cos \theta} \geq \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \geq \lim_{\theta \to 0} \cos \theta \rightarrow
\]

\[
\frac{1}{1} \geq \lim_{\theta \to 0} \frac{\sin \theta}{\theta} > 1 \rightarrow
\]

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{(similarly for } \theta < 0). \]