

Math 21A

Kouba

Derive the Product Rule Using the Limit definition of the Derivative

Let $F(x) = f(x)g(x)$. It's derivative is

$$\begin{aligned}F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\&= \lim_{h \rightarrow 0} \left[\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right] \\&= \lim_{h \rightarrow 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x) \right] \\&= f(x) \cdot g'(x) + f'(x) \cdot g(x) \\&= f(x) \cdot g'(x) + f'(x) \cdot g(x)\end{aligned}$$

i.e.,

$$D\{f(x)g(x)\} = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$