Math 21A (Fall 2017)
Kouba
Exam 2

Please PRINT your name here : KEY

Your Exam ID Number ________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE SOMEONE ELSE TAKE YOUR EXAM FOR YOU. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You will be graded on proper use of derivative notation.

7. You have until 3:00 p.m. sharp to finish the exam. Please close your exam immediately when time is called. Thank you.

1.) (5 pts. each) Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

a.) $y = 2x^2 - 3^x + 5^{100}$

$\xrightarrow{D}$ $y' = 4x - 3^x \ln 3 + 0$

b.) $f(x) = x \cdot \sin 2x$

$\xrightarrow{D}$ $f'(x) = x \cdot \cos 2x \cdot 2 + (1) \sin 2x$

c.) $g(x) = \ln(\sec^4(x^5))$

$\xrightarrow{D}$ $g'(x) = \dfrac{1}{\sec^4(x^5)} \cdot 4\sec^3(x^5) \cdot \sec(x^5)\tan(x^5) \cdot 5x^4$

d.) $y = e^x \cdot 7^{x^2} \cdot \log_3 x$ $\xrightarrow{D}$

$y' = (e^x) 7^{x^2} \log_3 x + e^x (7^{x^2} 2x \cdot \ln 7) \log_3 x + e^x 7^{x^2} \cdot \left(\frac{1}{x} \cdot \frac{1}{\ln 3}\right)$

e.) $y = \dfrac{\arcsin x}{x - arc\sec 4x}$

$\xrightarrow{D}$ $y' = \dfrac{(x - \operatorname{arcsec} 4x) \cdot \frac{1}{\sqrt{1-x^2}} - \arcsin x \cdot \left[1 - \frac{1}{|4x|\sqrt{(4x)^2 - 1}} \cdot (4)\right]}{(x - \operatorname{arcsec} 4x)^2}$

f.) $y = (\sin x)^x$ $\longrightarrow \ln y = \ln(\sin x)^x = x \ln(\sin x)$

$\xrightarrow{D}$ $\frac{1}{y} y' = x \cdot \frac{1}{\sin x} \cdot \cos x + (1) \ln(\sin x)$

$\longrightarrow$ $y' = (\sin x)^x \cdot \left[\frac{x \cos x}{\sin x} + \ln(\sin x)\right]$

2.) (7 pts.) Use the limit definition of the derivative to differentiate the function $f(x) = \sqrt{x^2+4}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+4} - \sqrt{x^2+4}}{h} \cdot \frac{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}}{\sqrt{(x+h)^2+4} + \sqrt{x^2+4}}$$

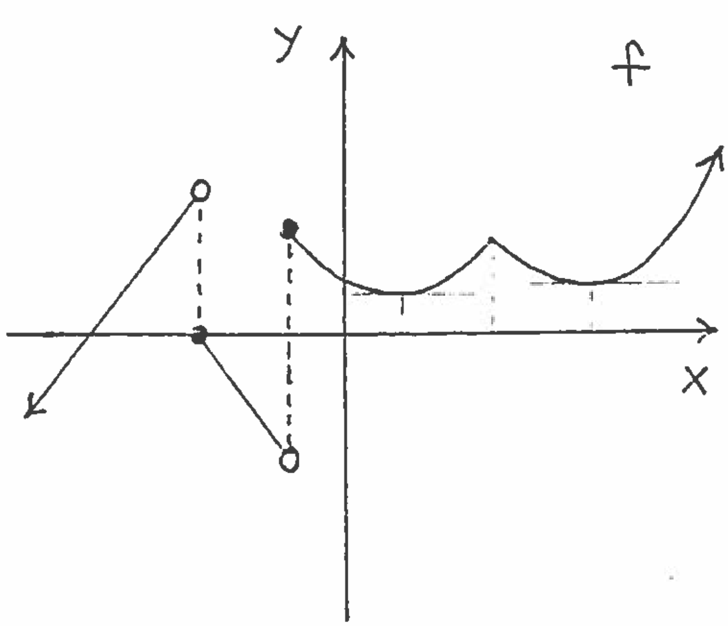
$$= \lim_{h\to 0} \frac{(\cancel{x^2} + 2hx + h^2 + \cancel{4}) - (\cancel{x^2} + \cancel{4})}{h\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)}$$

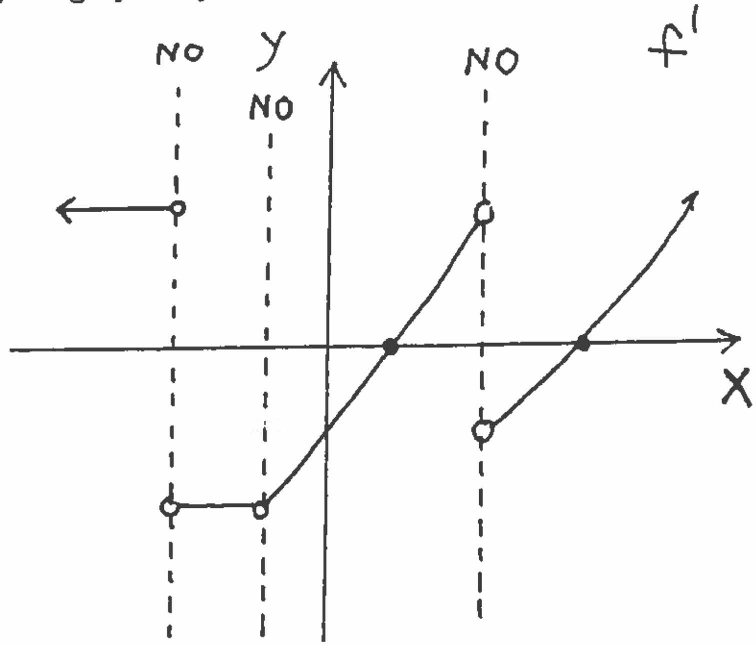
$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{(x+h)^2+4} + \sqrt{x^2+4}\right)}$$

$$= \frac{2x}{\sqrt{x^2+4} + \sqrt{x^2+4}} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}}$$

$$= \frac{x}{\sqrt{x^2+4}}$$

3.) (7 pts.) Sketch the graph of f' using the graph of f.





4.) (7 pts.) Assume that y is a function of x. Determine y' and y'' for $x^2 - e^y = y^2 + 3$ at $x = 2, y = 0$.

$$\xrightarrow{D} 2X - e^Y Y' = 2YY' \rightarrow 2X = e^Y Y' + 2YY' \rightarrow$$

$$2X = (e^Y + 2Y)Y' \rightarrow Y' = \frac{2X}{e^Y + 2Y} \quad (\text{Let } X=2, Y=0)$$

$$\rightarrow Y' = \frac{4}{1+0} = 4 \; ; \; \xrightarrow{D}$$

$$Y'' = \frac{(e^Y + 2Y)(2) - 2X(e^Y Y' + 2Y')}{(e^Y + 2Y)^2}$$

$$(\text{Let } X=2, Y=0, Y'=4) \rightarrow$$

$$Y'' = \frac{(1+0)2 - 4(4+8)}{(1+0)^2} \rightarrow Y'' = -46$$

5.) (5 pts. each) Let $f(x) = x(x-8)^3$.

a.) Solve $f'(x) = 0$ for x and set up a sign chart for f'.

$$\xrightarrow{D} f'(x) = x \cdot 3(x-8)^2 + (1)(x-8)^3$$

$$= (x-8)^2(3x + (x-8)) = (x-8)^2(4x-8) = 0$$

− 0 + 0 + f'

$X=2$ $X=8$

b.) Solve $f''(x) = 0$ for x and set up a sign chart for f''.

$$\xrightarrow{D} f''(x) = (x-8)^2(4) + 2(x-8)\cdot(4x-8)$$

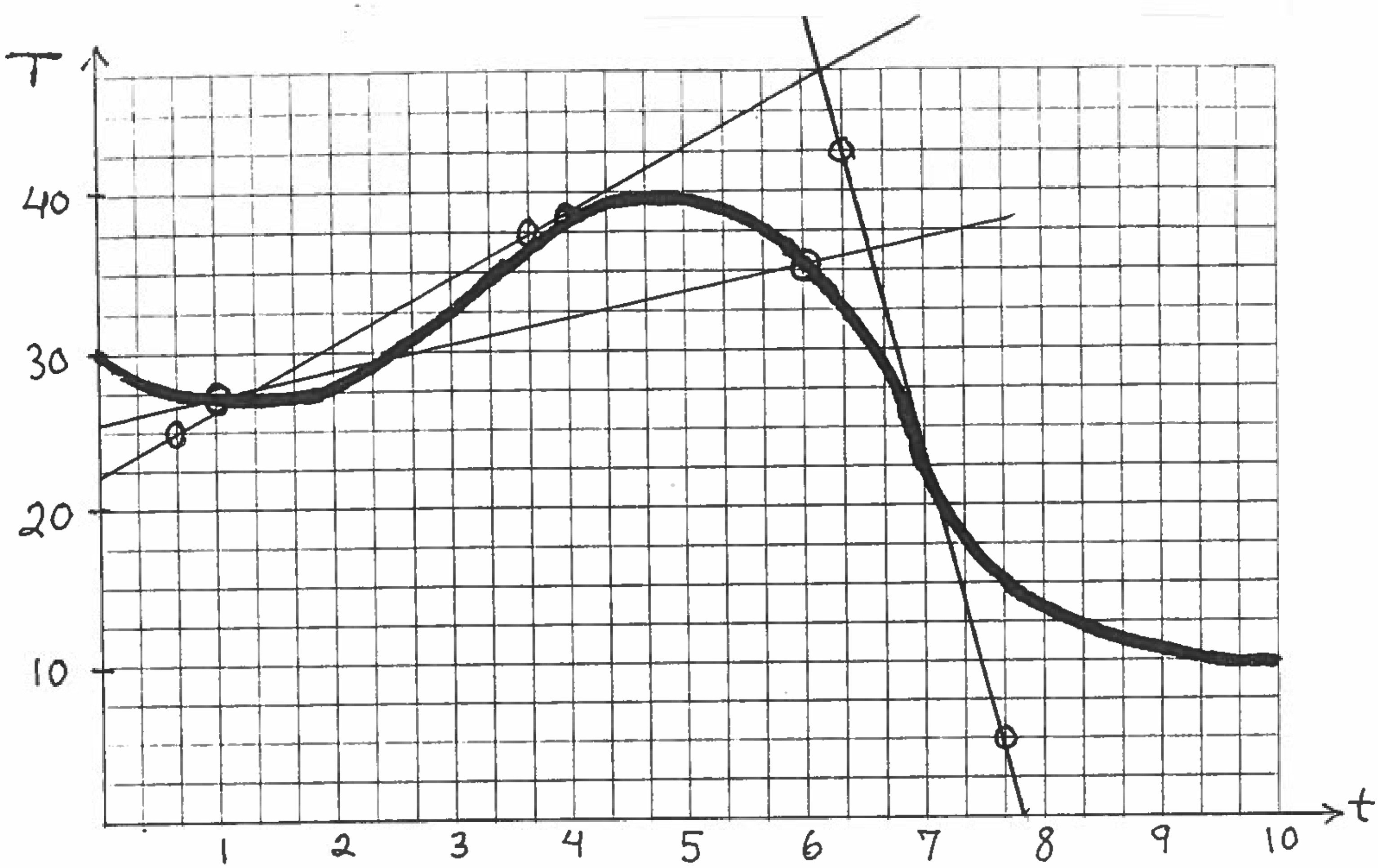
$$= 4(x-8)\left[(x-8) + 2(x-2)\right]$$

$$= 4(x-8)\left[3x - 12\right] = 0$$

+ 0 − 0 + f''

$X=4$ $X=8$

6.) Assume that the given graph represents the temperature $T(t)$ in $°C$ in a room after t hours.



a.) (4 pts.) Estimate the average rate of temperature change for $t = 1$ to $t = 6$ hours.

$$ARC = \frac{T(6) - T(1)}{6 - 1} \approx \frac{36 - 27.5}{5} = 1.7 \ °C/hr.$$

b.) (4 pts.) Estimate the instantaneous rate of temperature change for $t = 4$ hours.

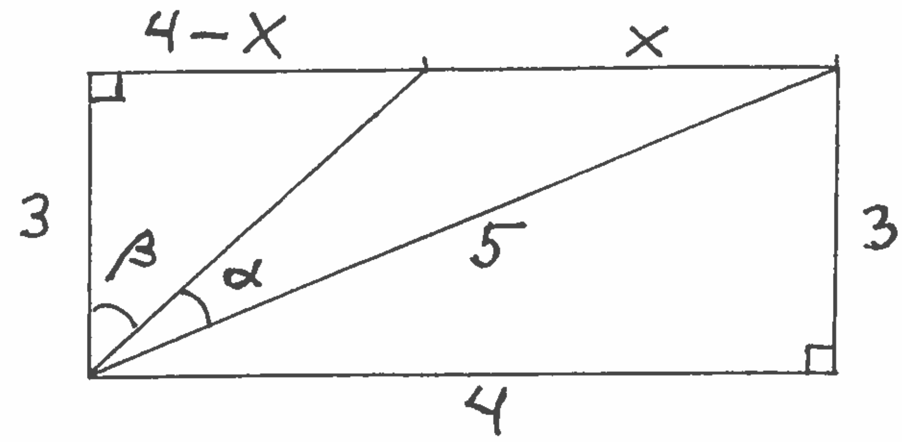
$$IRC = \frac{rise}{run} \approx \frac{37.5 - 25}{3} \approx 4.17 \ °C/hr.$$

c.) (4 pts.) Estimate the specific time when the instantaneous rate of temperature change is smallest. What is your estimate for the instantaneous rate of temperature change at this time ?

i.) at $t \approx 7$ hrs.

ii.) $IRC = \frac{rise}{run} \approx \frac{42.5 - 5}{6\frac{1}{3} - 7\frac{2}{3}} = {}^{-}28.125 \ °C/hr.$

7.) (6 pts.) Consider the given diagram. Write α as a function of x.

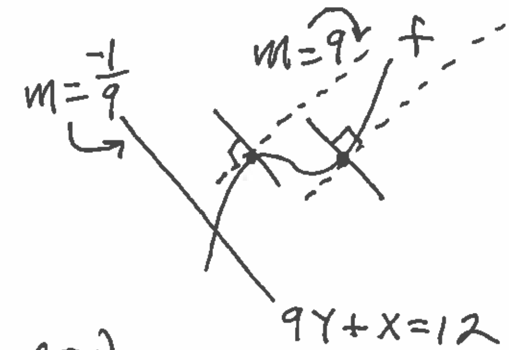


$\beta = \arctan\left(\frac{4-X}{3}\right)$ and

$\alpha + \beta = \arctan(4/3) \rightarrow$

$\alpha = \arctan(4/3) - \arctan\left(\frac{4-X}{3}\right)$

8.) (8 pts.) Find all points (x, y) on the graph of $f(x) = x^3 + 3x^2$ with normal ($\perp$) lines parallel to the line $9y + x = 12$.



$9Y + X = 12 \rightarrow 9Y = -X + 12$

$\rightarrow Y = \frac{-1}{9}X + \frac{4}{3}$ so $m = \frac{-1}{9}$;

$f(X) = X^3 + 3X^2 \xrightarrow{D}$

$f'(X) = 3X^2 + 6X = 9$ ($\perp$ SLOPE)

$\rightarrow 3X^2 + 6X - 9 = 0$

$\rightarrow 3(X^2 + 2X - 3) = 0$

$\rightarrow 3(X-1)(X+3) = 0$

$\rightarrow$ $X = 1, Y = 4$ or $X = -3, Y = 0$

9.) (7 pts.) Differentiate the following function and SIMPLIFY your answer as much as possible : $f(x) = \arctan(e^x) + arc\cot(e^{-x})$.

$$\xrightarrow{D} f'(x) = \frac{1}{1+(e^x)^2} \cdot e^x + \frac{-1}{1+(e^{-x})^2} \cdot -e^{-x}$$

$$= \frac{e^x}{1+e^{2x}} + \frac{1/e^x}{1+1/e^{2x}} \cdot \frac{e^{2x}}{e^{2x}}$$

$$= \frac{e^x}{1+e^{2x}} + \frac{e^x}{e^{2x}+1} = \frac{2e^x}{1+e^{2x}}$$

10.) (6 pts.) Use the limit definition of the derivative to show that $f(x) = x^{2/3}$ is NOT differentaible at $x = 0$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^{2/3}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h^{1/3}} = \frac{1}{0} = \pm\infty \quad (\text{so D.N.E.})$$

The following is an EXTRA CREDIT PROBLEM. This problem is OPTIONAL.

1.) (8 pts.) Prove that $\log_b x^m = m \log_b x$.

Let $\log_b x = z \rightarrow x = b^z$, then

$$\log_b x^m = \log_b (b^z)^m$$

$$= \log_b b^{mz}$$

$$= mz$$

$$= m \log_b x$$