Math 21A (Fall 2017)
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER PERSON TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You will be graded on proper use of derivative and limit notation.

7. You have until 3:00 p.m. sharp to finish the exam.

1.) (6 pts. each) Use L'Hopital's Rule to evaluate the following limits.

a.) $\lim_{x\to -1} \dfrac{x^4+x}{x^{12}-1}$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{X\to -1} \frac{4X^3+1}{12X^{11}} = \frac{-4+1}{-12} = \frac{-3}{-12} = \frac{1}{4}$$

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b.) $\lim_{x\to 0} (x+1)^{1/x^2} \overset{\text{"}1^{\infty}\text{"}}{=} \lim_{X\to 0^-} e^{\ln (X+1)^{1/X^2}}$

$$= e^{\lim_{X\to 0^-} \frac{\ln(X+1)}{X^2}} \overset{\text{"}\frac{0}{0}\text{"}}{=} e^{\lim_{X\to 0^-} \frac{\frac{1}{X+1}}{\frac{2X}{1}}}$$

$$= e^{\lim_{X\to 0^-} \frac{1}{(X+1)2X}} = e^{\frac{1}{0^-}} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

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c.) $\lim_{x\to\infty} x^3 e^{-2x} = \text{"}\infty\cdot 0\text{"} = \lim_{X\to\infty} \dfrac{X^3}{e^{2X}}$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{X\to\infty} \frac{3X^2}{2e^{2X}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{X\to\infty} \frac{6X}{4e^{2X}}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{X\to\infty} \frac{6}{8e^{2X}} = \frac{6}{\infty} = 0$$

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d.) $\lim_{x\to 1} \dfrac{(\ln x)^2}{x^2-2x+1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{X\to 1} \dfrac{2\ln X\cdot \frac{1}{X}}{2X-2}$

$$= \lim_{X\to 1} \frac{\cancel{2}\ln X}{\cancel{2}X(X-1)} = \lim_{X\to 1} \frac{\ln X}{X^2-X}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{X\to 1} \frac{\frac{1}{X}}{2X-1} = \frac{1}{1} = 1$$

2.) (8 pts.) Use a differential to show that $\ln(1+3h^2) \approx 3h^2$ for small h.

Let $f(x) = \ln x$ and $x : 1 \to 1+3h^2$,

so $\Delta x = (1+3h^2) - 1 = 3h^2$; $f'(x) = \frac{1}{x}$;

$\Delta f = f(1+3h^2) - f(1) = \ln(1+3h^2) - \overset{0}{\cancel{\ln 1}}$

$= \ln(1+3h^2)$;

$df = f'(1) \cdot \Delta x = \frac{1}{(1)} \cdot 3h^2 = 3h^2$; then

$\Delta f \approx df \rightarrow$

$\ln(1+3h^2) \approx 3h^2$

3.) (8 pts.) Assume that the derivative of function f is $f'(x) = (x-2)^3(x-7)^2$. Determine all x-values corresponding to inflection points for f.

$\xrightarrow{D} f''(x) = (x-2)^3 \cdot 2(x-7) + 3(x-2)^2 \cdot (x-7)^2$

$= (x-7)(x-2)^2 \cdot [2(x-2) + 3(x-7)]$

$= (x-7)(x-2)^2 \cdot [5x - 25] = 0$

$+ \quad 0 \quad + \quad 0 \quad - \quad 0 \quad +$ $\quad f''$

$x=2 \qquad x=5 \qquad x=7$

Infl. Pts.

4.) (8 pts.) Assume that $\lim_{x\to 1} \frac{ax^2+bx}{x^2+3x-4} = \frac{4}{5}$. Determine the values of constants a and b.

$$\lim_{x\to 1} \frac{ax^2+bx}{x^2+3x-4} = \frac{a+b}{0} \overset{\text{"}\frac{0}{0}\text{"}}{=} \frac{4}{5}, \text{ so}$$

$a+b=0 \rightarrow \boxed{a=-b}$; then

$$\lim_{x\to 1} \frac{-bx^2+bx}{x^2+3x-4} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{-2bx+b}{2x+3}$$

$$= \frac{-2b+b}{5} = \frac{-b}{5} = \frac{4}{5} \rightarrow -b=4 \rightarrow$$

$b=-4$, $a=4$

5.) (8 pts.) Determine if the following function satisfies the assumptions of the MVT on the given closed interval. If so, find all values of c guaranteed by the conclusion of the MVT.

$f(x) = x + \ln x$ on $[1, e]$

f is cont. on $[1,e]$ since it is the sum of cont. functions $y=x$ (line) and $y=\ln x$ (well-known); $\xrightarrow{D}$

$f'(x) = 1 + \frac{1}{x}$ so f is diff. on $(1,e)$; then there is at least one value c, $1<c<e$, so that $f'(c) = \frac{f(e)-f(1)}{e-1}$

$$\rightarrow 1+\frac{1}{c} = \frac{e+\overset{1}{\cancel{\ln e}}-(1+\overset{0}{\cancel{\ln 1}})}{e-1} = \frac{e}{e-1} \rightarrow$$

$$\frac{1}{c} = \frac{e}{e-1} - \frac{e-1}{e-1} = \frac{1}{e-1} \rightarrow \boxed{c=e-1}$$

6.) (12 pts.) Consider the function $f(x) = x - 12x^{1/3}$ on the closed interval $[-1, 27]$. Determine the open intervals where f is increasing ($\uparrow$), decreasing ($\downarrow$), concave up ($\cup$), and concave down ($\cap$). Identify all relative and absolute extrema, inflection points, and x- and y-intercepts. Sketch the graph.

The first and second derivatives are $f'(x) = \dfrac{x^{2/3} - 4}{x^{2/3}}$ and $f''(x) = \dfrac{8}{3x^{5/3}}$.

$\xrightarrow{D} f'(x) = \dfrac{x^{2/3} - 4}{x^{2/3}} = 0 \rightarrow x^{2/3} = 4 \rightarrow x^2 = 4^3 = 64$

$\rightarrow x = \pm 8 = +8 \; , \; x \neq 0$

f'		−	NO	−	0	+	
	$x=-1$		$x=0$		$x=8$		$x=27$
	$\underbrace{Y=11}_{\text{abs. max.}}$		$Y=0$		$\underbrace{Y=-16}_{\text{abs. min}}$		$\underbrace{Y=-9}_{\text{rel. max.}}$

$\xrightarrow{D} f''(x) = \dfrac{8}{3x^{5/3}} = 0$

f''		−	NO	+	
	$x=-1$		$x=0$		$x=27$
			$\underbrace{Y=0}_{\text{infl. pt.}}$		

$x=0: \; Y=0$

$Y=0: \; 0 = x - 12x^{1/3}$

$= x^{1/3}(x^{2/3} - 12)$

$\rightarrow x=0, \; x^{2/3} = 12$

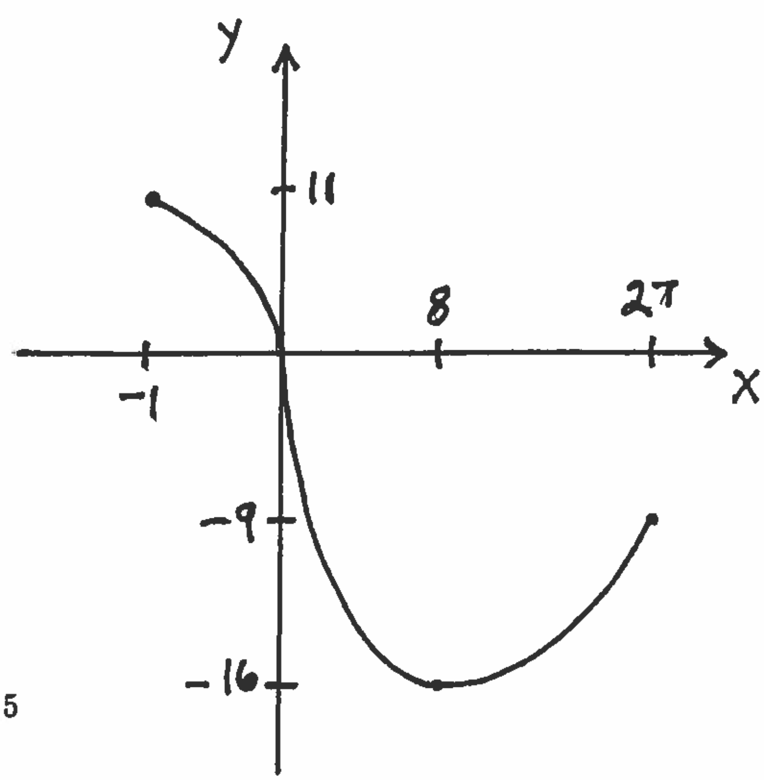
$\approx \pm 41.6$

f is $\uparrow$ for $8 < x < 27$,

f is $\downarrow$ for $-1 < x < 0$, $0 < x < 8$,

f is $\cup$ for $0 < x < 27$,

f is $\cap$ for $-1 < x < 0$.



7.) (8 pts.) Use a differential to estimate the value of $\sqrt{27}$.

Let $f(x)=\sqrt{x}$ and $x: 25 \to 27$, so

$\Delta x = 27-25 = 2$; $f'(x) = \dfrac{1}{2\sqrt{x}}$;

$\Delta f = f(27) - f(25) = \sqrt{27} - \sqrt{25} = \sqrt{27} - 5$;

$df = f'(25) \cdot \Delta x$

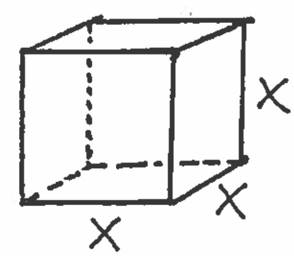
$= \dfrac{1}{2\sqrt{25}}(2) = \dfrac{1}{5} = 0.2$; then

$\Delta f \approx df \to$

$\sqrt{27} - 5 \approx 0.2 \longrightarrow \sqrt{27} \approx 5.2$

8.) (8 pts.) Consider a solid cube of edge length x. Assume that the cube's volume is measured with an absolute percentage error of at most 9%. Use a differential to estimate the maximum absolute percentage error in computing the cube's edge length x.

$V = x^3$ and given $\dfrac{|\Delta V|}{V} \le 9\%$, find $\dfrac{|\Delta x|}{x}$:



$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta x|}{V}$$

$$= \frac{|3x^2 \cdot \Delta x|}{x^3} = 3 \cdot \frac{|\Delta x|}{x} \le 9\%$$

$$\longrightarrow \quad \frac{|\Delta x|}{x} \le 3\%$$

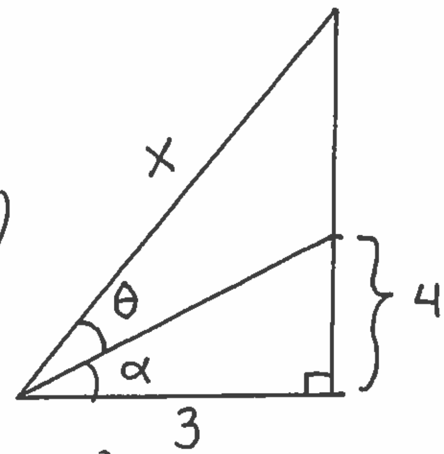
9.) (8 pts.) Consider the given triangle. If θ increases at the rate of 3 radians per hour, at what rate is x changing when $x = 6$?

Given $\frac{d\theta}{dt} = 3$ rad./hr. ;

Find $\frac{dx}{dt}$ when $x = 6$;

$\tan\alpha = 4/3 \to \alpha = \arctan(4/3)$

$\cos(\theta+\alpha) = \frac{3}{x} \to$

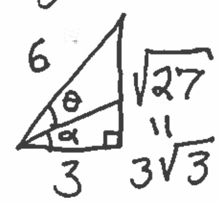


$$x = \frac{3}{\cos(\theta+\alpha)} \to x = 3\sec(\theta+\alpha) \xrightarrow{D}$$

$$\frac{dx}{dt} = 3\sec(\theta+\alpha)\tan(\theta+\alpha)\left(\frac{d\theta}{dt} + \cancelto{0}{\frac{d\alpha}{dt}}\right)$$

(Let $x = 6$, SEE DIAGRAM)

$$= \cancel{3}\left(\frac{6}{\cancel{3}}\right)\left(\frac{\cancel{3}\sqrt{3}}{\cancel{3}}\right)\cdot 3 = 18\sqrt{3} \text{ units/hr.}$$

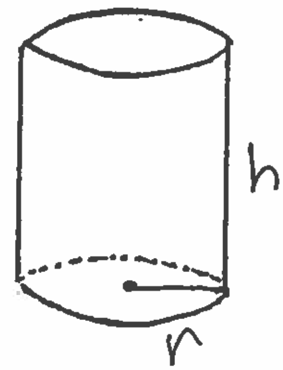


10.) (8 pts.) Consider a closed circular cylinder of radius r and height h. If the radius r is increasing at the rate of 5 ft./min. and the height h is decreasing at the rate of 2 ft./min. Determine the rate at which the cylinder's surface area is changing when $r = 3$ ft. and $h = 4$ ft.

$S = 2\pi r^2 + 2\pi r h$, given

$\frac{dr}{dt} = 5$ ft./min., $\frac{dh}{dt} = -2$ ft./min.,

find $\frac{dS}{dt}$ when $r = 3$, $h = 4$:



$$\frac{dS}{dt} = 4\pi r \cdot \frac{dr}{dt} + 2\pi r \cdot \frac{dh}{dt} + 2\pi \frac{dr}{dt}\cdot h$$

$$= 4\pi(3)(5) + 2\pi(3)(-2) + 2\pi(5)(4)$$

$$= 60\pi - 12\pi + 40\pi$$

$$= 88\pi \text{ ft.}^2/\text{min.}$$

9.) OR $\theta+\alpha = \arccos\left(\frac{3}{x}\right) \xrightarrow{D}$

$$\frac{d\theta}{dt} + \cancel{\frac{d\alpha}{dt}}^{\,0} = \frac{-1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \cdot \frac{-3}{x^2} \cdot \frac{dx}{dt} \longrightarrow$$

(Let $x=6$)

$$1\cancel{3} = \frac{-1}{\sqrt{3/4}} \cdot \frac{-\cancel{3}}{36} \cdot \frac{dx}{dt} \longrightarrow$$

$$\frac{dx}{dt} = \frac{\sqrt{3}}{2} \cdot 36 = 18\sqrt{3} \text{ units/hr.}$$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Consider the function $f(x) = \begin{cases} 0 \ , & \text{if } x = 0 \\ x^2 \sin(1/x) \ , & \text{if } x \neq 0 \end{cases}$.

a.) Is f continuous at $x = 0$? Verify your answer. $\boxed{\text{YES}}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0 = f(0)$$

since $-1 \leq \sin\left(\frac{1}{x}\right) \leq +1 \rightarrow$

$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$, and

$\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2$, so

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by

Squeeze Principle

b.) Is f differentiable at $x = 0$? Verify your answer. $\boxed{\text{YES}}$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

since $-1 \leq \sin\left(\frac{1}{h}\right) \leq +1 \rightarrow$ $(h > 0)$

$-h \leq h \sin\left(\frac{1}{h}\right) \leq h$, and

$\lim_{h \to 0} -h = 0 = \lim_{h \to 0} h$, so

$\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$

by Squeeze Principle

same for $(h < 0)$