Math 21A
Kouba
Exam 1

PRACTICE EXAM 1

KEY

Please PRINT your name here : ____________________

Your HW/Exam ID Number ______

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 7 pages, including the cover page.

6. You may NOT use L'Hopital's Rule on this exam.

7. You may NOT use the shortcuts for finding limits to infinity.

8. Using only a calculator to determine limits will receive little or no credit.

9. You will be graded on proper use of limit notation.

10. You have until 9:40 a.m. sharp to finish the exam.

1.) (7 pts. each) Determine the following limits.

a.) $\lim_{x\to 1} \frac{x^2-x}{x^2+x-2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 1} \frac{x\cancel{(x-1)}}{(x+2)\cancel{(x-1)}} = \frac{1}{3}$

b.) $\lim_{x\to 3} \frac{\frac{1}{3}-\frac{1}{x}}{\frac{x-3}{1}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to 3} \frac{\cancel{x-3}}{3x}\cdot\frac{1}{\cancel{x-3}} = \frac{1}{9}$

c.) $\lim_{h\to 0} \frac{\sin 5h}{2h} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{h\to 0} \frac{1}{2}\cdot 5\cdot\frac{\sin 5h}{5h} = \frac{1}{2}\cdot 5\cdot 1 = \frac{5}{2}$

d.) $\lim_{x\to -3^+} \frac{x^2-12}{x+3} = \frac{-3}{0^+} = -\infty$

-2.9

$X = -3$

e.) $\lim_{x\to -1} \frac{2-\sqrt{x+5}}{x+1}\cdot\frac{2+\sqrt{X+5}}{2+\sqrt{X+5}} = \lim_{X\to -1} \frac{4-(X+5)}{(X+1)(2+\sqrt{X+5})}$

$= \lim_{X\to -1} \frac{-\cancel{(X+1)}}{\cancel{(X+1)}(2+\sqrt{X+5})} = \frac{-1}{2+2} = \frac{-1}{4}$

2.) (7 pts.) Use the Squeeze Principle to evaluate the following limit :

$\lim_{x\to\infty} \frac{3x+\cos^2(3x+1)}{7-4x}$; $-1 \le \cos(3x+1) \le +1 \to 0 \le \cos^2(3x+1) \le 1$

$\to \frac{0}{x} \le \frac{\cos^2(3x+1)}{x} \le \frac{1}{x}$ and $\lim_{x\to\infty} \frac{0}{x} = 0 = \lim_{x\to\infty}\frac{1}{x}$;

so by Squeeze Principle $\lim_{x\to\infty} \frac{\cos^2(3x+1)}{x} = 0$; then

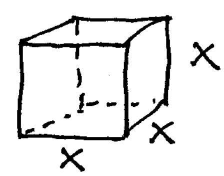
$$\lim_{x\to\infty} \frac{3x+\cos^2(3x+1)}{7-4x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3+\frac{\cos^2(3x+1)}{x}}{\frac{7}{x}-4} = \frac{3+0}{0-4} = -\frac{3}{4}$$

3.) (8 pts.) Determine the domain and range for $f(x) = 5 + \sqrt{2x-8}$.

$2x-8 \ge 0 \to 2x \ge 8 \to$ Domain : $x \ge 4$;

since $0 \le 2x-8 < \infty$ and $\lim_{x\to\infty}(2x-8) = \infty \to$

Range : $Y \ge 5$



4.) Consider a three-dimensional cube with side length x.

a.) (2 pts.) Write the volume V of the cube as a function of x.

Volume $V = (\text{length})(\text{width})(\text{height})$
$= x\cdot x\cdot x = x^3 \to V = x^3$

b.) (2 pts.) Write the surface area S of the cube as a function of x.

Surface area $S = 6\cdot x^2$

c.) (4 pts.) Write the surface area S of the cube as a function of the volume V.

$S = 6x^2$ and $V = x^3 \to x = V^{1/3}$, so that

$S = 6(V^{1/3})^2 \to S = 6\cdot V^{2/3}$

5.) Consider the following function $f(x) = \begin{cases} \dfrac{x^2-2x}{x^2-4}, & \text{if } x \neq 2, -2 \\ \frac{1}{2}, & \text{if } x = 2 \\ 0, & \text{if } x = -2. \end{cases}$

a.) (4 pts.) Determine if f is continuous at $x = 2$.

i.) $f(2) = 1/2$

ii.) $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2-2x}{x^2-4} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x \to 2} \dfrac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$

$= \dfrac{2}{4} = 1/2$

iii) $\lim_{x \to 2} f(x) = \frac{1}{2} = f(2)$, so f <u>is</u> continuous at $x = 2$.

b.) (4 pts.) Determine if f is continuous at $x = -2$.

i.) $f(-2) = 0$

ii.) $\lim_{x \to -2} f(x) = \lim_{x \to -2} \dfrac{x^2-2x}{x^2-4} = \dfrac{8}{0^{\pm}} = \pm\infty$,

so f is <u>not</u> continuous at $x = -2$.

c.) (4 pts.) For what x-values is f continuous ? Explain.

$Y = x^2 - 2x$ and $Y = x^2 - 4$ are continuous for all values of x (since both are polynomials). Thus $Y = \dfrac{x^2-2x}{x^2-4}$ is continuous (quotient of continuous functions) for <u>all</u> x-values <u>except</u> where $x^2 - 4 = (x-2)(x+2) = 0$, i.e., except $x = 2$, $x = -2$. So f is continuous for <u>all x-values except $x = -2$</u>.

6.) (10 pts.) Use the Intermediate Value Theorem to show that the equation $x^3 = 4 + \sqrt{x}$ is solvable. This is a writing exercise.

Since $x^3 - \sqrt{x} - 4 = 0$, let $\boxed{f(x) = x^3 - \sqrt{x} - 4}$, which is $\boxed{\text{continuous}}$ for $x \geq 0$ since it is the difference of continuous functions, and let $\boxed{m = 0}$. Since $f(0) = -4$ and $f(4) = 58$, and $m = 0$ is $\boxed{\text{between}}$ these values, use the interval $\boxed{[0, 4]}$. By the IMVT it follows that there is at least one number c, $0 \leq c \leq 4$, so that $f(c) = m$, i.e.,

$$c^3 - \sqrt{c} - 4 = 0,$$

and equation is solvable.

7.) (10 pts.) Give an ε, δ-proof for the following limit. This is a writing exercise. : $\lim\limits_{x \to -1} (3x^2 + 5) = 8$

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x - (-1)| < \delta$, then $|f(x) - 8| < \varepsilon$, i.e., if $0 < |x+1| < \delta$, then $|f(x) - 8| < \varepsilon$. Begin with $|f(x) - 8| < \varepsilon$ and solve for $|x+1|$. Then

$|f(x) - 8| < \varepsilon$ iff $|(3x^2 + 5) - 8| < \varepsilon$

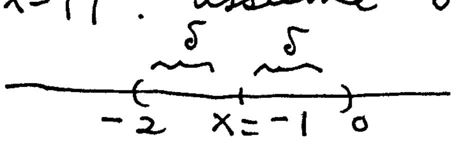
iff $|3x^2 - 3| < \varepsilon$

iff $3|x^2 - 1| < \varepsilon$

iff $3|(x-1)(x+1)| < \varepsilon$

iff $3|x-1||x+1| < \varepsilon$.

Keep $|x+1|$ and "replace" $|x-1|$. assume $\delta \leq 1$



so that $-2 < x < 0$ and $1 < |x-1| < 3$. Then

$3|x-1||x+1| < 3(3)|x+1| < \varepsilon$

iff $9|x+1| < \varepsilon$ iff $|x+1| < \varepsilon/9$.

Choose $\delta = \min\{1, \varepsilon/9\}$. Thus, if $0 < |x+1| < \delta$, it follows that $|f(x) - 8| < \varepsilon$.

QED

8.) (10 pts.) Use intercepts and horizontal, vertical, and tilted asymptotes to sketch a graph of $y = \dfrac{x^3 - 1}{x^2 - 4}$.

$X = 0$: $\underline{Y = \frac{1}{4}}$; $Y = 0$: $X^3 - 1 = 0 \rightarrow \underline{X = 1}$;

$$\left.\begin{array}{l} \lim\limits_{X \to 2^+} Y = \text{"}\dfrac{7}{0^+}\text{"} = +\infty \\ \lim\limits_{X \to 2^-} Y = \text{"}\dfrac{7}{0^-}\text{"} = -\infty \end{array}\right\} \text{V.A.} : \boxed{X = 2}$$

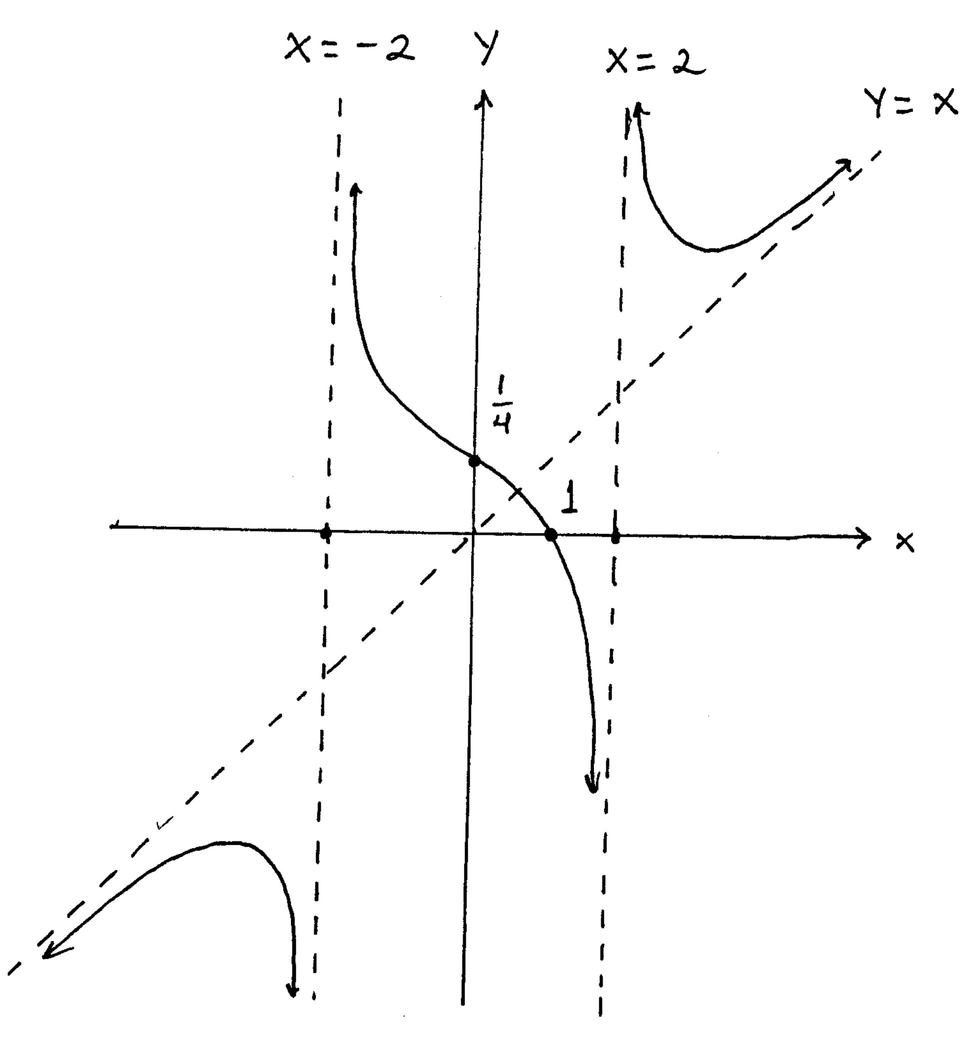
$$\left.\begin{array}{l} \lim\limits_{X \to -2^+} Y = \text{"}\dfrac{-9}{0^-}\text{"} = +\infty \\ \lim\limits_{X \to -2^-} Y = \text{"}\dfrac{-9}{0^+}\text{"} = -\infty \end{array}\right\} \text{V.A.} : \boxed{X = -2}$$

$$\left.\begin{array}{l} \lim\limits_{X \to \infty} \dfrac{X^3 - 1}{X^2 - 4} \cdot \dfrac{\frac{1}{X^2}}{\frac{1}{X^2}} = \lim\limits_{X \to \infty} \dfrac{X - \cancelto{0}{\frac{1}{X^2}}}{1 - \cancelto{0}{\frac{4}{X^2}}} = +\infty \\ \lim\limits_{X \to -\infty} \dfrac{X^3 - 1}{X^2 - 4} = \lim\limits_{X \to -\infty} \dfrac{X - \cancelto{0}{\frac{1}{X^2}}}{1 - \cancelto{0}{\frac{4}{X^2}}} = -\infty \end{array}\right\} \text{No H.A.}$$

$$\begin{array}{r} X \\ X^2 - 4 \,\overline{)\, X^3 - 1 } \\ -(X^3 - 4X) \\ \hline 4X - 1 \end{array} \quad \text{so}$$

$$Y = \frac{X^3 - 1}{X^2 - 4} = X + \frac{4X - 1}{X^2 - 4}$$

so that $\boxed{Y = X}$ is <u>tilted</u> <u>asymptote</u>.



The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Determine the domain for $f(x) = \dfrac{555}{3 - \sqrt{x^2 - 8x}}$.

$x^2 - 8x = x(x-8) \geq 0$

$+ \quad 0 \quad - \quad 0 \quad +$

$x = 0 \qquad x = 8$

and

$3 - \sqrt{x^2 - 8x} \neq 0 \rightarrow$

$3 \neq \sqrt{x^2 - 8x} \rightarrow$

$9 \neq x^2 - 8x \rightarrow$

$0 \neq x^2 - 8x - 9 = (x-9)(x+1) \rightarrow$

$x \neq 9, \quad x \neq -1$

<u>Domain</u>: $x \geq 8, \; x \leq 0$ but $x \neq 9, \; x \neq -1$.