

Math 21B (Kouba)

Comparison Tests for Improper Integrals Absolute Convergence Test

Comparison Test for Convergence of

Assume f and g are continuous functions with $0 \leq f(x) \leq g(x)$. If $\int_a^\infty g(x) dx$ converges (finite), then $\int_a^\infty f(x) dx$ converges.

$$\begin{aligned} \text{Ex: } \int_2^\infty \frac{2x+1}{x^5+x^4+3} dx &\leq \int_2^\infty \frac{2x+x}{x^5+0+0} dx \\ &= \int_2^\infty \frac{3x}{x^5} dx = 3 \int_2^\infty \frac{1}{x^4} dx = 3 \cdot \lim_{A \rightarrow \infty} \int_2^A x^{-4} dx \\ &= 3 \cdot \lim_{A \rightarrow \infty} \left. \frac{-1}{3x^3} \right|_2^A = 3 \cdot \lim_{A \rightarrow \infty} \left\{ \frac{-1}{3A^3} - \frac{-1}{24} \right\} = \frac{1}{24} < \infty, \end{aligned}$$

so $\int_2^\infty \frac{2x+1}{x^5+x^4+3} dx$ converges.

Comparison Test for Divergence:

Assume f and g are continuous functions with $0 \leq g(x) \leq f(x)$. If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ diverges.

$$\begin{aligned} \text{Ex: } \int_1^\infty \frac{x^3+3}{x^4+1} dx &\geq \int_1^\infty \frac{x^3+0}{x^4+x^4} dx \\ &= \int_1^\infty \frac{x^3}{2x^4} dx = \frac{1}{2} \int_1^\infty \frac{1}{x} dx = \frac{1}{2} \lim_{A \rightarrow \infty} \int_1^A \frac{1}{x} dx \\ &= \frac{1}{2} \lim_{A \rightarrow \infty} \ln|x| \Big|_1^A = \frac{1}{2} \lim_{A \rightarrow \infty} (\ln A - \ln 1) = \infty, \end{aligned}$$

so $\int_1^\infty \frac{x^3+3}{x^4+1} dx$ diverges.

Absolute Convergence Test:

Assume f is any continuous function with positive or negative values.

If $\int_a^{\infty} |f(x)| dx$ converges, then

$$\int_a^{\infty} f(x) dx \text{ converges.}$$

Ex: Consider $\int_3^{\infty} \frac{\sin^3 7x}{x^2+9} dx$; since

$$\int_3^{\infty} \left| \frac{\sin^3 7x}{x^2+9} \right| dx = \int_3^{\infty} \frac{|\sin^3 7x|}{x^2+9} dx$$

$$= \int_3^{\infty} \frac{|\sin 7x|^3}{x^2+9} dx \leq \int_3^{\infty} \frac{(1)^3}{x^2+3^2} dx$$

$$= \lim_{A \rightarrow \infty} \int_3^A \frac{1}{x^2+3^2} dx = \lim_{A \rightarrow \infty} \frac{1}{3} \arctan\left(\frac{x}{3}\right) \Big|_3^A$$

$$= \lim_{A \rightarrow \infty} \left(\frac{1}{3} \arctan\left(\frac{A}{3}\right) - \frac{1}{3} \arctan 1 \right)$$

$$= \lim_{A \rightarrow \infty} \left(\frac{1}{3} \arctan\left(\frac{A}{3}\right) - \frac{1}{3} \cdot \frac{\pi}{4} \right)$$

$$= \frac{1}{3} \cdot \arctan(\infty) - \frac{\pi}{12} = \frac{1}{3} \cdot \left(\frac{\pi}{2}\right) - \frac{\pi}{12} = \frac{\pi}{12} < \infty,$$

it follows that $\int_3^{\infty} \left| \frac{\sin^3 7x}{x^2+9} \right| dx$ converges

by the Comparison Test so that

$$\int_3^{\infty} \frac{\sin^3 7x}{x^2+9} dx \text{ converges by the}$$

Absolute Convergence Test.