Math 21B (Kouba)
Comparison Test for Convergence and Divergence
Absolute Convergence Test

1.) Use the Comparison Test or Absolute Convergence Test to show that each of the following improper integrals converges.

a.) $\displaystyle\int_0^\infty \frac{\sin 3x}{x^2+1}\,dx$

b.) $\displaystyle\int_1^\infty \frac{2}{x^{11}+x^{10}}\,dx$

c.) $\displaystyle\int_2^\infty \frac{\cos(x^2)}{x^2+\cos^2 x}\,dx$

d.) $\displaystyle\int_3^\infty \frac{e^{-x^2}}{x^3}\,dx$

e.) $\displaystyle\int_0^1 \frac{e^{-1/x^2}}{x^2}\,dx$

f.) $\displaystyle\int_0^\infty \frac{e^{-1/x^2}}{x^2}\,dx$

2.) Use the Comparison Test to show that each of the following improper integrals diverges.

a.) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x}}\,dx$

b.) $\displaystyle\int_0^1 \frac{1}{x+x^5}\,dx$

Comparison Tests

1.) a.) $\int_0^\infty \left|\frac{\sin 3x}{x^2+1}\right| dx = \int_0^\infty \frac{|\sin 3x|}{x^2+1}\, dx$

$\leq \int_0^\infty \frac{1}{x^2+1}\, dx = \lim_{A\to\infty} \int_0^A \frac{1}{x^2+1}\, dx$

$= \lim_{A\to\infty} \arctan x \Big|_0^A = \lim_{A\to\infty} (\arctan A - \cancelto{0}{\arctan 0})$

$= \frac{\pi}{2}$; since $\int_0^\infty \left|\frac{\sin 3x}{x^2+1}\right| dx < \infty$ then

$\int_0^\infty \frac{\sin 3x}{x^2+1}\, dx < \infty$ by Absolute Convergence Test.

b.) $\int_1^\infty \frac{2}{x^{11}+x^{10}}\, dx \leq \int_1^\infty \frac{2}{x^{10}+x^{10}}\, dx$

$= \int_1^\infty \frac{2}{2x^{10}}\, dx = \lim_{A\to\infty} \int_1^A \frac{1}{x^{10}}\, dx = \lim_{A\to\infty} \int_1^A x^{-10}\, dx$

$= \lim_{A\to\infty} \frac{x^{-9}}{-9}\Big|_1^A = \lim_{A\to\infty} \frac{-1}{9}\left(\cancelto{0}{\frac{1}{A^9}} - \frac{1}{1^9}\right) = \frac{1}{9}$ so

$\int_1^\infty \frac{2}{x^{11}+x^{10}}\, dx < \infty$.

c.) $\int_2^\infty \left|\frac{\cos(x^2)}{x^2+\cos^2 x}\right| dx = \int_2^\infty \frac{|\cos(x^2)|}{x^2+\cos^2 x}\, dx$

$\leq \int_2^\infty \frac{1}{x^2+\cos^2 x}\, dx \leq \int_2^\infty \frac{1}{x^2+0}\, dx$

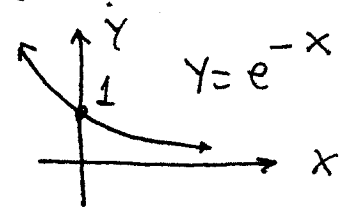
$= \lim_{A\to\infty} \int_2^A x^{-2}\, dx = \lim_{A\to\infty} \frac{-1}{x}\Big|_2^A$

$= \lim_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{2}\right) = \frac{1}{2}$; since

$\int_2^\infty \left|\frac{\cos(x^2)}{x^2+\cos^2 x}\right| dx < \infty$ then $\int_2^\infty \frac{\cos(x^2)}{x^2+\cos^2 x}\, dx < \infty$

by Absolute Convergence Test

d.) $\displaystyle\int_3^\infty \frac{e^{-x^2}}{x^3}\,dx \le \int_3^\infty \frac{1}{x^3}\,dx$



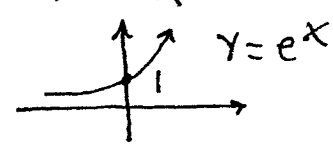
$\displaystyle= \lim_{A\to\infty}\int_3^A x^{-3}\,dx = \lim_{A\to\infty} \frac{x^{-2}}{-2}\Big|_3^A$

$\displaystyle= \lim_{A\to\infty} -\frac{1}{2}\left(\cancel{\frac{1}{A^2}}^{\,0} - \frac{1}{3^2}\right) = \frac{1}{18}$ so $\displaystyle\int_3^\infty \frac{e^{-x^2}}{x^3}\,dx < \infty.$

e.) $\displaystyle\int_0^1 \frac{e^{-1/x^2}}{x^2}\,dx = \lim_{A\to 0^+}\int_A^1 \frac{e^{-1/x^2}}{x^2}\,dx$

(For $0 < x \le 1 \to x > x^2 \to \frac{1}{x} < \frac{1}{x^2} \to$

$\displaystyle\frac{-1}{x^2} < \frac{-1}{x} \to e^{-1/x^2} < e^{-1/x}$



)

$\displaystyle\le \lim_{A\to 0^+}\int_A^1 \frac{e^{-1/x}}{x^2}\,dx$

$\displaystyle= \lim_{A\to 0^+} e^{-1/x}\Big|_A^1 = \lim_{A\to 0^+}\left(e^{-1} - e^{-1/A}\right)$

$\displaystyle= \frac{1}{e} - e^{-\infty} = \frac{1}{e} - \frac{1}{e^\infty} = \frac{1}{e} - 0 = \frac{1}{e}$ so

$\displaystyle\int_0^1 \frac{e^{-1/x^2}}{x^2}\,dx < \infty$

f.) $\displaystyle\int_0^\infty \frac{e^{-1/x^2}}{x^2}\,dx = \int_0^1 \frac{e^{-1/x^2}}{x^2}\,dx + \int_1^\infty \frac{e^{-1/x^2}}{x^2}\,dx$

$= C + D$; C converges by part e.);

$\displaystyle D = \int_1^\infty \frac{e^{-1/x^2}}{x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx = \lim_{A\to\infty}\int_1^A x^{-2}\,dx$

$$= \lim_{A\to\infty} \left.\frac{-1}{x}\right|_A^A = \lim_{A\to\infty} \left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{1}\right) = 1 \text{ so}$$

D converges; thus $\int_0^\infty \frac{e^{-1/x^2}}{x^2}\,dx < \infty$.

2.) a.) $$\int_1^\infty \frac{1}{\sqrt{x^2+x}}\,dx \geq \int_1^\infty \frac{1}{\sqrt{x^2+x^2}}\,dx$$

$$= \lim_{A\to\infty} \int_1^A \frac{1}{\sqrt{2}\cdot x}\,dx = \lim_{A\to\infty} \left.\frac{1}{\sqrt{2}} \ln x \right|_1^A$$

$$= \lim_{A\to\infty} \frac{1}{\sqrt{2}}\left(\ln A - \cancelto{0}{\ln 1}\right) = \infty \quad \text{so}$$

$\int_1^\infty \frac{1}{\sqrt{x^2+x}}\,dx$ diverges.

b.) $$\int_0^1 \frac{1}{x+x^5}\,dx = \lim_{A\to 0^+} \int_A^1 \frac{1}{x+x^5}\,dx$$

$$\geq \lim_{A\to 0^+} \int_A^1 \frac{1}{x+x}\,dx \quad (\text{since } x > x^5 \text{ for } 0 < x \leq 1)$$

$$= \lim_{A\to 0^+} \int_A^1 \frac{1}{2x}\,dx = \lim_{A\to 0^+} \left.\frac{1}{2}\ln x\right|_A^1$$

$$= \lim_{A\to 0^+} \frac{1}{2}\left(\cancelto{0}{\ln 1} - \ln A\right) = \frac{1}{2} - (-\infty) = \infty$$

so $\int_0^1 \frac{1}{x+x^5}\,dx$ diverges.