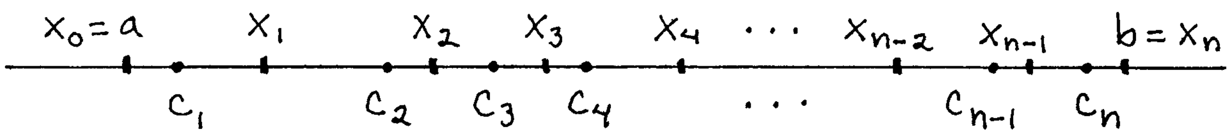
Math 21B
Kouba
Limit Definition of the Definite Integral

1.) Consider a function $y = f(x)$ defined on the closed interval $[a, b]$.

2.) *Partition* the interval $[a, b]$ into n pieces of any size :

$$x_0 = a,\ x_1,\ x_2,\ x_3, \cdots,\ x_{n-1},\ b = x_n$$

Define the *mesh* of the partition to be : $\max\limits_{1 \le i \le n} (x_i - x_{i-1})$. (In other words, the mesh of a partition is the length of the largest subinterval.)



3.) Pick *sampling points* $c_1,\ c_2,\ c_3, \cdots,\ c_{n-1},\ c_n$, where c_i is a number in the subinterval $[x_{i-1},\ x_i]$ for $i = 1, 2, 3, \cdots, n$. Let $\Delta x_i = x_i - x_{i-1}$ be the length of the subinterval $[x_{i-1},\ x_i]$ for $i = 1, 2, 3, \cdots, n$.

4.) Compute the *Riemann Sum* defined by $\displaystyle\sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$

5.) The *Definite Integral* is then given to be

$$\int_a^b f(x)\ dx = \lim_{mesh \to 0} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$$

REMARK : The above is a formal and general definition of the Definite Integral. When doing problems using this step-by-step process for computing $\displaystyle\int_a^b f(x)\ dx$, it is convenient to

i.) pick equally-spaced partition points so that all of the subintervals have the same length : $\Delta x_i = \dfrac{b-a}{n}$ for $i = 1, 2, 3, \cdots, n$;

ii.) pick sampling points to be the right-hand endpoints of the subintervals : $c_i = a + \dfrac{b-a}{n} \cdot i$ for $i = 1, 2, 3, \cdots, n$; then

iii.) $\displaystyle\int_a^b f(x)\ dx = \lim_{mesh \to 0} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i$.