Math 21B
Kouba
MVT and FTC's

Mean – Value Theorem for Integrals : If f is a continuous function on the closed interval $[a, b]$, then there is at least one number c , $a \le c \le b$, so that

$$f(c)(b-a) = \int_a^b f(x)\,dx \ .$$

Proof : Since f is a continuous function on the closed interval $[a, b]$, by the Maximum- and Minimum-Value Theorems (pp. 79-80), f has a maximum vaule M and a minimum value m on $[a, b]$, i.e., $m \le f(x) \le M$ on $[a, b]$. By property 7.) (p. 291) of definite integrals,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \ ,$$

so that

$$m \le \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{call this number } y_o} \le M \ ,$$

By the Intermediate-Value Theorem (p. 81) there is at least one number c , $a \le c \le b$, so that

$$f(c) = y_o, \text{ i.e., } f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \ ,$$

so that

$$f(c)(b-a) = \int_a^b f(x)\,dx \ .$$

First Fundamental Theorem of Calculus (FTC1) : Assume that f is a continuous function on the closed interval $[a, b]$ and that $F(x) = \displaystyle\int_a^x f(t)\,dt$. Then $F'(x) = f(x)$.

Proof : Consider $F(x) = \displaystyle\int_a^x f(t)\,dt$ as the area under the graph of f above the interval $[a, x]$. Then $F(x+h)$ is the area under the graph of f above the interval $[a, x+h]$ and $F(x+h) - F(x)$ is the area of the " thin strip " from x to $x+h$, i.e., $F(x+h) - F(x) = \displaystyle\int_x^{x+h} f(t)\,dt$. By the Mean-Value Theorem for integrals there is at least one number c , $x \le c \le x+h$, so that

$$f(c) \cdot h = \int_x^{x+h} f(t)\,dt$$

The derivative of $F(x)$ can now be computed as

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{\displaystyle\int_x^{x+h} f(t)\,dt}{h} \\
&= \lim_{h\to 0} \frac{f(c)\ h}{h} \\
&= \lim_{h\to 0} f(c) \qquad \text{(Recall that } x \le c \le x+h.\text{)} \\
&= f(x)\ .
\end{aligned}$$

Second Fundamental Theorem of Calculus (FTC2) : Let f be a continuous function on the closed interval $[a,b]$. Assume that $F(x)$ is an antiderivative of $f(x)$, i.e., assume that $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)\ .$$

Proof : Let $A(x) = \displaystyle\int_a^x f(t)\,dt$. Then $A(a) = 0$, $A(b) = \displaystyle\int_a^b f(t)\,dt$, and $A'(x) = f(x)$ by FTC1. But $F'(x) = f(x)$. By Corollary 2 (p. 168) to the Mean-Value Theorem $F(x) = A(x) + C$ for any constant C, or

$$A(x) = F(x) - C\ .$$

Then

$$\begin{aligned}
&\int_a^b f(x)\,dx = \int_a^b f(t)\,dt \\
&= A(b) \\
&= A(b) - A(a) \\
&= (F(b) - C) - (F(a) - C) \\
&= F(b) - F(a) \\
&= F(x)\Big|_a^b\ .
\end{aligned}$$