Math 21C
Kouba
Alternating Series Test (For Convergence Only)

DEFINITION : A series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots ,$$

where $a_n > 0$ for all values of $n = 1, 2, 3, 4, \cdots$, is called an *alternating series* . How can we test this series for convergence ? We will need the following fact, which is given without proof.

FACT A : Assume that the sequence $\{b_n\}$ satisfies the following two conditions :

1.) $b_1 < b_2 < b_3 < b_4 < \cdots$, i.e., $b_n < b_{n+1}$ for $n = 1, 2, 3, 4, \cdots$ (The sequence is strictly increasing.) and

2.) $b_n < C$, a fixed constant, for $n = 1, 2, 3, 4, \cdots$ (The sequence is bounded.).

Then $\displaystyle\lim_{n\to\infty} b_n = L$ for some finite number L.

Alternating Series Test : Consider the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. If the sequence $\{a_n\}$ is positive (+), decreasing ($\downarrow$), and $\displaystyle\lim_{n\to\infty} a_n = 0$, then the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof : Use the sequence of partial sums (even and odd separately). Since $\{a_n\}$ is positive and decreasing, the following must be true :

$$s_2 = a_1 + (-a_2) < a_1 ,$$

$$s_4 = a_1 + \underbrace{(-a_2) + a_3}_{(-)} + \underbrace{(-a_4)}_{(-)} < a_1 ,$$

$$s_6 = a_1 + \underbrace{(-a_2) + a_3}_{(-)} + \underbrace{(-a_4) + a_5}_{(-)} + \underbrace{(-a_6)}_{(-)} < a_1 , \cdots$$

and

$$s_{2n} = a_1 + \underbrace{(-a_2) + a_3}_{(-)} + \underbrace{(-a_4) + a_5}_{(-)} + \cdots + \underbrace{(-a_{2n-2}) + a_{2n-1}}_{(-)} + \underbrace{(-a_{2n})}_{(-)} < a_1 , \cdots ;$$

also

$$s_2 = \underbrace{(a_1 - a_2)}_{(+)} ,$$

$$s_4 = (a_1 - a_2) + (a_3 - a_4) = s_2 + \underbrace{(a_3 - a_4)}_{(+)} > s_2 ,$$

$$s_6 = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) = s_4 + \underbrace{(a_5 - a_6)}_{(+)} > s_4 , \cdots$$

and

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) = s_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{(+)} > s_{2n-2}\ , \ \cdots$$

thus, the sequence of even partial sums $\{s_{2n}\} = \{s_2, s_4, s_6, s_8, \cdots\}$ is increasing and bounded above by a_1. It follows from FACT A that

$$\lim_{n\to\infty} s_{2n} = L\ , \text{ for some finite number } L.$$

Now consider the sequence of odd partial sums $\{s_{2n-1}\} = \{s_1, s_3, s_5, s_7, \cdots\}$. Note that

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} \\ &= (a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1}) - a_{2n} \\ &= s_{2n-1} - a_{2n}\ , \end{aligned}$$

i.e.,

$$s_{2n-1} = s_{2n} + a_{2n}\ .$$

Taking the limit of both sides we get

$$\lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n} \ + \ \lim_{n\to\infty} a_{2n} = L + 0 = L\ .$$

Now the sequence of *all* partial sums $\{s_n\} = \{s_1, s_2, s_3, s_4, s_5, \cdots\}$ satisfies $\lim_{n\to\infty} s_n = L$.

Thus,

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \lim_{n\to\infty} s_n = L$$

and the alternating series converges by the sequence of partial sums test.