Math 21C
Kouba
Exact Change, Differential, Chain Rule

Assume that function \( z = f(x, y) \) has continuous partial derivatives and that point \((x, y)\) changes from \((x_1, y_1)\) to \((x_2, y_2)\). Let \( z_1 = f(x_1, y_1) \) and \( z_2 = f(x_2, y_2) \). Define the exact change in \( f \) (or \( z \)) to be

\[
\Delta f = z_2 - z_1.
\]

Let \( \Delta x = x_2 - x_1 \) and \( \Delta y = y_2 - y_1 \). Now define the differential of \( f \) (or \( z \)) to be

\[
df = \frac{\partial f}{\partial x}(x_1, y_1) \Delta x + \frac{\partial f}{\partial y}(x_1, y_1) \Delta y.
\]

It can be proven using continuity, the Mean Value Theorem, and the differential for a function of one variable that

\[
\Delta f = \frac{\partial f}{\partial x}(x_1, y_1) \Delta x + \frac{\partial f}{\partial y}(x_1, y_1) \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,
\]

where \( \epsilon_1 \to 0 \) and \( \epsilon_2 \to 0 \) as \( \Delta x \to 0 \) and \( \Delta y \to 0 \). It follows immediately that

\[
\Delta f \approx df
\]

if both \( \Delta x \) and \( \Delta y \) are “small.” Thus, the differential \( df \) can be considered an approximation to the exact change \( \Delta f \).

Now assume that \( z = f(x, y), x = g(t), \) and \( y = h(t) \). The above equation for \( \Delta f \) leads to the following chain rule:

\[
\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

If \( z = f(x, y), x = g(u, v), \) and \( y = h(u, v) \), then the above equation for \( \Delta f \) leads to the following chain rules:

\[
\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.
\]