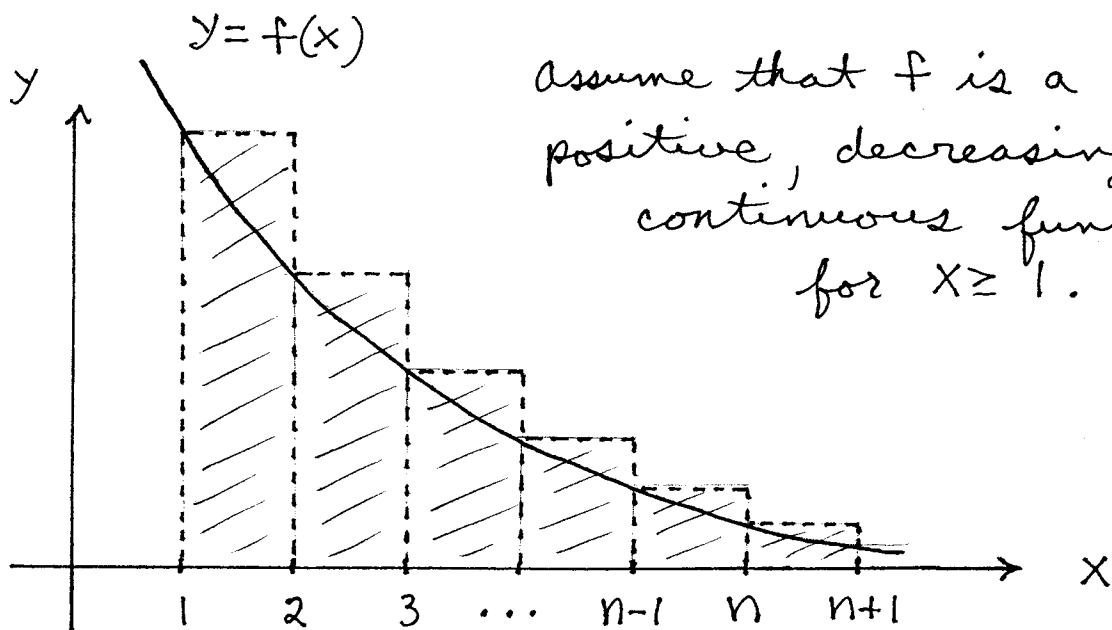


Math 21C

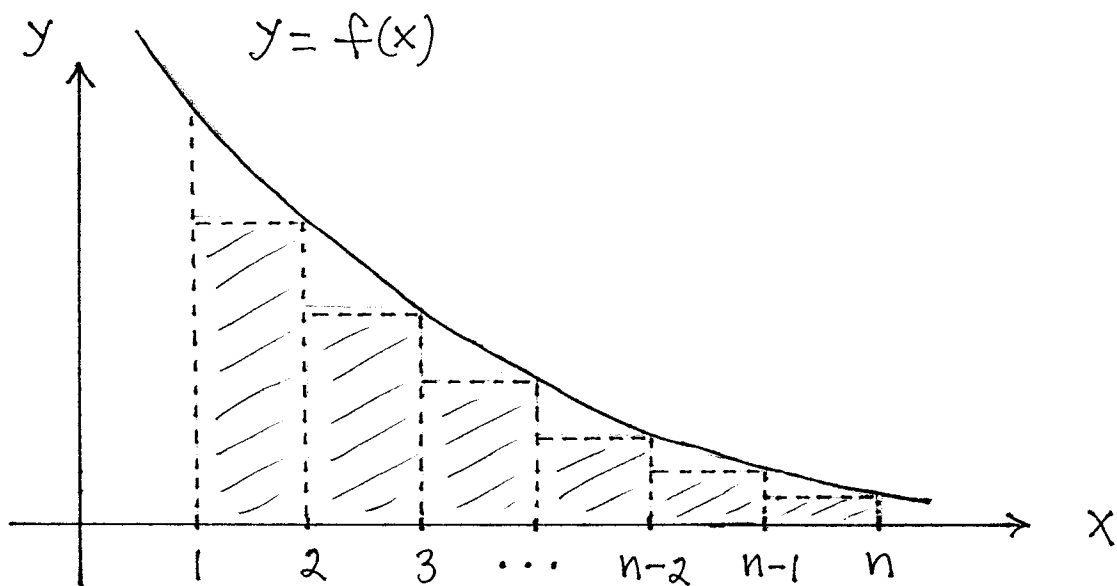
Kouba

# Infinite Series and Improper Integrals



assume that  $f$  is a  
positive, decreasing,  
continuous function  
for  $x \geq 1$ .

$$\int_1^{n+1} f(x) dx < f(1) + f(2) + f(3) + \dots + f(n) ;$$



$$f(2) + f(3) + \dots + f(n) < \int_1^n f(x) dx \Rightarrow$$

$$f(1) + f(2) + f(3) + \dots + f(n) < f(1) + \int_1^n f(x) dx ;$$

It follows that

$$(*) \quad \int_1^{n+1} f(x) dx < f(1) + f(2) + \dots + f(n) < f(1) + \int_1^n f(x) dx.$$

Case 1: Assume that  $\int_1^{\infty} f(x) dx = L$  (finite). Then

$$\begin{aligned} f(1) + f(2) + \dots + f(n) &< f(1) + \int_1^n f(x) dx \Rightarrow \\ \lim_{n \rightarrow \infty} (f(1) + f(2) + \dots + f(n)) &< \lim_{n \rightarrow \infty} (f(1) + \int_1^n f(x) dx) \Rightarrow \\ \sum_{n=1}^{\infty} f(n) &< f(1) + \int_1^{\infty} f(x) dx = f(1) + L < \infty, \text{ i.e.,} \\ \sum_{n=1}^{\infty} f(n) &\text{ converges.} \end{aligned}$$

Case 2: Assume that  $\int_1^{\infty} f(x) dx = \infty$ . Then

$$\begin{aligned} \int_1^{n+1} f(x) dx &< f(1) + f(2) + \dots + f(n) \Rightarrow \\ \lim_{n \rightarrow \infty} \int_1^{n+1} f(x) dx &\leq \lim_{n \rightarrow \infty} (f(1) + f(2) + \dots + f(n)) \Rightarrow \\ \int_1^{\infty} f(x) dx &\leq \sum_{n=1}^{\infty} f(n) \Rightarrow \sum_{n=1}^{\infty} f(n) = \infty, \text{ i.e.,} \\ \sum_{n=1}^{\infty} f(n) &\text{ diverges.} \end{aligned}$$

This verifies the following series test.

Integral Test: Assume that function  $f$  is positive, decreasing, and continuous for  $x \geq 1$ .

a.) If  $\int_1^{\infty} f(x) dx$  converges, then  $\sum_{n=1}^{\infty} f(n)$  converges.

b.) If  $\int_1^{\infty} f(x) dx$  diverges, then  $\sum_{n=1}^{\infty} f(n)$  diverges.

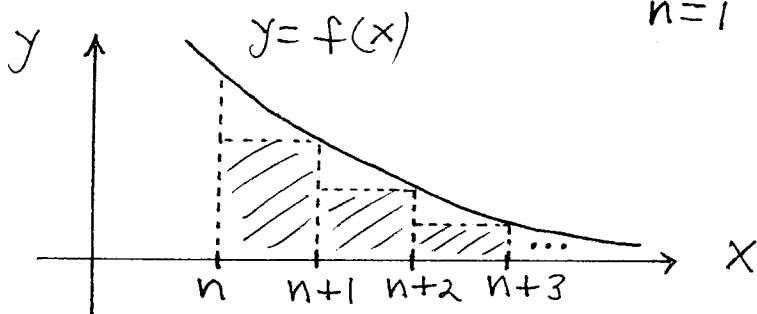
Note that

$$\sum_{n=1}^{\infty} f(n) = \underbrace{f(1) + f(2) + \dots + f(n)}_{S_n} + \underbrace{f(n+1) + f(n+2) + \dots}_{R_n}$$

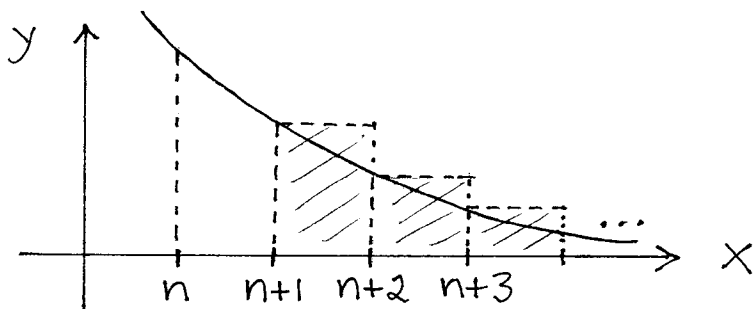
let  $S_n = f(1) + f(2) + \dots + f(n)$  be the  $n$ th partial sum, and let the infinite tail

$$R_n = f(n+1) + f(n+2) + \dots$$

be the error or remainder term for the infinite series  $\sum_{n=1}^{\infty} f(n)$ . Then



$$f(n+1) + f(n+2) + f(n+3) + \dots < \int_n^{\infty} f(x) dx, \quad \text{and}$$



$\int_{n+1}^{\infty} f(x) dx < f(n+1) + f(n+2) + \dots$ , so that

(\*)(\*) 
$$\int_{n+1}^{\infty} f(x) dx < f(n+1) + f(n+2) + \dots < \int_n^{\infty} f(x) dx$$