Math 21C
Kouba

The Lagrange Form of the Remainder for the Taylor Series

**Question**: For what x-values is a function $y = f(x)$ equal to its Taylor Series centered at $x = a$?

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$R_n(x; a) = f(x) - P_n(x; a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(c)}{(n+2)!}(x-a)^{n+2} + \ldots$$

$P_n(x; a)$: Taylor polynomial of degree $n$

$R_n(x; a)$: Taylor remainder (error)

$$f(x) = P_n(x; a) + R_n(x; a) \Rightarrow$$

$$\lim_{n \to \infty} f(x) = \lim_{n \to \infty} P_n(x; a) + \lim_{n \to \infty} R_n(x; a) \Rightarrow$$

$$f(x) = \text{(Taylor series)} + (0) \Rightarrow$$

**Answer**: It must be those x-values for which $\lim_{n \to \infty} R_n(x; a) = 0$. 
Fact: (Lagrange Form of Taylor Remainder)

\[ R_n(x; a) = \frac{f^{(n+1)}(c_n) \cdot (x-a)^{n+1}}{(n+1)!}, \]  

where \( c_n \) is a number between \( a \) and \( x \).

Example: Show that \( e^x \) is equal to its Maclaurin series for all values of \( x \):

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots; \]

\[ f^{(n)}(x) = e^x \quad \text{for } n = 0, 1, 2, 3, \ldots \text{ then for any value of } x \]

\[ |R_n(x; 0)| = \left| \frac{f^{(n+1)}(c_n) \cdot (x-0)^{n+1}}{(n+1)!} \right| \]

\[ = e^{c_n} \cdot \frac{|x|^{n+1}}{(n+1)!}, \]  

where \( c_n \) is between 0 and \( x \)

\[ \begin{align*}
1: & \quad \frac{|x|^{n+1}}{(n+1)!} \quad \text{if } x < 0 \\
\text{or} & \quad e^x \cdot \frac{|x|^{n+1}}{(n+1)!} \quad \text{if } x > 0 
\end{align*} \]

Since \( \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \) and

\[ \lim_{n \to \infty} e^x \cdot \frac{|x|^{n+1}}{(n+1)!} = e^x (0) = 0 \]

it follows that \( \lim_{n \to \infty} |R_n(x; 0)| = 0 \) so that \( \lim_{n \to \infty} R_n(x; 0) = 0 \).

Thus, \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \) for all values of \( x \).