Math 21C
Kouba
Discussion Sheet 11

1.) Recall that if $y = f(x)$ is a function and

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots = \sum_{n=0}^{\infty} a_n(x-a)^n$$

is the Taylor Series (or Maclaurin series if $a = 0$) centered at $x = a$ for $y = f(x)$, then $a_n = \dfrac{f^{(n)}(a)}{n!}$. Use this formula to compute the first four nonzero terms and the general formula for the Taylor series expansion for each function about the given value of a .

a.) $f(x) = e^x$ centered at $x = 0$ b.) $f(x) = e^x$ centered at $x = \ln 2$

c.) $f(x) = \dfrac{1}{1-x}$ centered at $x = 0$ d.) $f(x) = \sin x$ centered at $x = 0$

e.) $f(x) = \dfrac{1}{x}$ centered at $x = 1$ f.) $f(x) = \sqrt{x+5}$ centered at $x = -1$

2.) Use the suggested method to find the first four nonzero terms of the Maclaurin series for each function.

a.) $f(x) = \dfrac{1}{1+x^2}$ (Substitute $-x^2$ into the Maclaurin series for $\dfrac{1}{1-x}$.)

b.) $f(x) = x^3 e^{-3x}$ (Substitute $-3x$ into the Maclaurin series for e^x and then multiply by x^3 .)

c.) $f(x) = \dfrac{e^x}{1-x} = e^x \dfrac{1}{1-x}$ (Multiply the Maclaurin series for e^x and $\dfrac{1}{1-x}$ term by term and then group like powers of x .)

d.) $f(x) = \dfrac{e^x}{1-x}$ (Use polynomial division. Divide the Maclaurin series for e^x by $1-x$.)

e.) $f(x) = 3x^2\cos(x^3)$ (Substitute x^3 into the Maclaurin series for $\sin x$ then differentiate term by term.)

f.) $f(x) = \arctan x$ (Integrate the Maclaurin series for $\dfrac{1}{1+t^2}$ from 0 to x.)

3.) The Maclaurin series for $f(x) = \dfrac{1}{1+x}$ is $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

a.) Show that $f(x) = \dfrac{1}{1+x}$ and $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ have the same value at $x = 0$.

b.) Show that $f(x) = \dfrac{1}{1+x}$ and $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ have the same value at $x = 1/2$.

c.) Show that $f(x) = \dfrac{1}{1+x}$ and $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ do not have the same value at $x = 1$.

d.) For what x-values is $f(x) = \dfrac{1}{1+x}$ defined ?

e.) For what x-values is the Maclaurin series $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ defined ?

NOTE : It can be shown that $f(x) = \dfrac{1}{1+x}$ and its Maclaurin series $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ are equal on the interval $(-1, 1)$.

4.) The following definite integral cannot be evaluated using the Fundamental Theorem of calculus. Use the Maclaurin series for $\cos x$ and the absloute error $|R_n|$ for an alternating series to estimate the value of this integral with error at most 0.0001 : $\displaystyle\int_0^1 \cos(x^2)\,dx$

5.) Write each Maclaurin series as an ordinary function.

a.) $(3x) - \dfrac{(3x)^3}{3!} + \dfrac{(3x)^5}{5!} - \dfrac{(3x)^7}{7!} + \dfrac{(3x)^9}{9!} - \cdots$ (HINT: Use $\sin x$.)

b.) $x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$ (HINT: Factor.)

c.) $\dfrac{1}{2!} + \dfrac{x}{3!} + \dfrac{x^2}{4!} + \dfrac{x^3}{5!} + \dfrac{x^4}{6!} + \cdots$ (HINT: Use e^x.)

d.) $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots$ (Challenging)

6.) (Challenging) Consider the function $f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0 \;, & \text{if } x = 0 \;. \end{cases}$

a.) Determine $\lim\limits_{x \to \infty} f(x)$.

b.) Determine $\lim\limits_{x \to -\infty} f(x)$.

c.) Determine $\lim\limits_{x \to 0} f(x)$.

d.) Use a graphing calculator to graph this function.

e.) Compute $f'(0)$. You need to use the limit definition of the derivative.

f.) Compute $f''(0)$. You need to use the limit definition of the derivative.

g.) It can be shown that $f^{(n)}(0) = 0$ for $n = 0, 1, 2, 3, 4, \ldots$.

h.) Determine the Maclaurin series for $f(x)$.

i.) For what x-values is $f(x)$ defined ?

j.) For what x-values is its Maclaurin defined ?

k.) For what x-values are $f(x)$ and its Maclaurin series equal ?

"In mathematics you don't understand things. You just get used to them." – Johann von Neumann