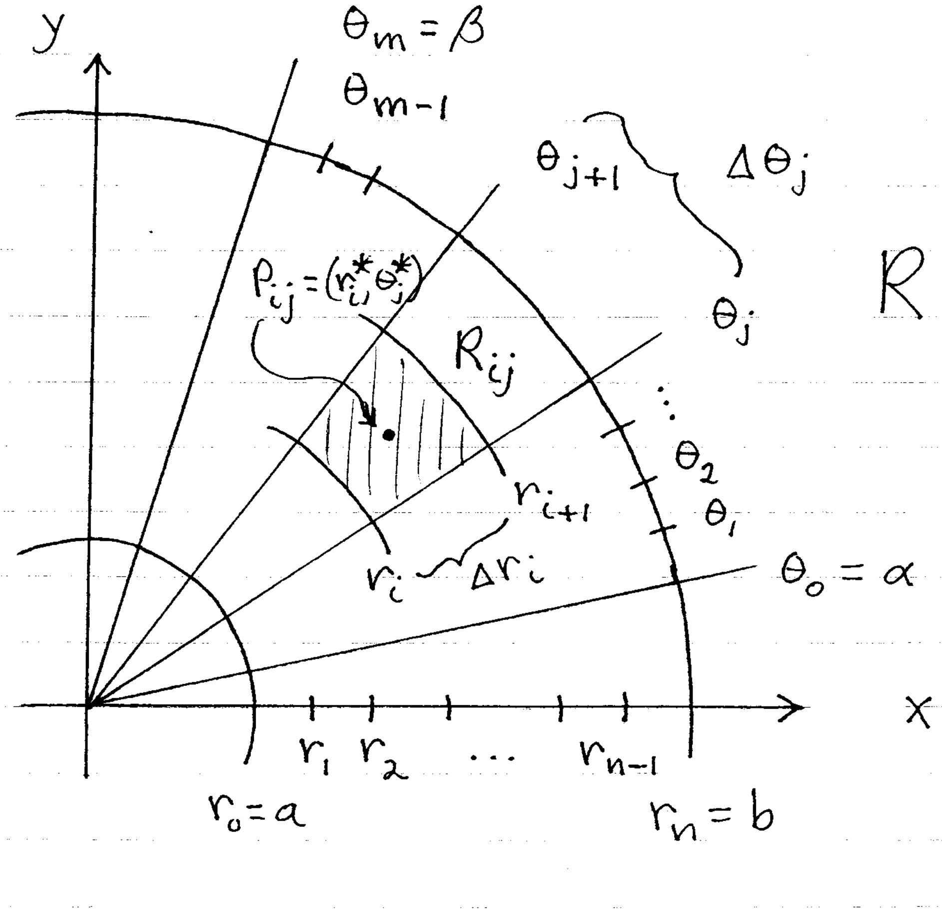
Math 21C
Koruba
Double Integration Using
Polar Coordinates

Assume that surface $z = f(P)$, where $P = (x, y)$, lies above the flat region R in the xy-plane and is given in polar form by $\alpha \le \theta \le \beta$ and $a \le r \le b$. Since $x = r\cos\theta$ and $y = r\sin\theta$ we can say that $z = f(P) = f(r, \theta)$. We seek to find the volume of the resulting solid using polar coordinates.

Begin by dividing the interval $[a, b]$ into n parts and the interval $[\alpha, \beta]$ into m parts.



This partitions region R into mn parts R_{ij} for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, m$. Let A_{ij} be the area of R_{ij}. Then

$$A_{ij} = (\text{fraction of circle})(\text{difference of areas of circles})$$

$$= \frac{\Delta\theta_j}{2\pi}\left[\pi r_{i+1}^2 - \pi r_i^2\right]$$

$$= \frac{\Delta\theta_j}{2\cancel{\pi}} \cdot \cancel{\pi}(r_{i+1} + r_i)(r_{i+1} - r_i)$$

$$= \left(\frac{r_{i+1} + r_i}{2}\right)(\Delta r_i)(\Delta\theta_j)$$

$$= r_i^* (\Delta r_i)(\Delta\theta_j), \text{ where } r_i^* = \frac{r_{i+1} + r_i}{2}.$$

Choose sampling point $P_{ij} = (r_i^*, \theta_j^*)$, where $r_i^* = \frac{1}{2}(r_{i+1} + r_i)$ and $\theta_j \le \theta_j^* \le \theta_{j+1}$.

The sum $\sum_{i=1}^{n} f(P_{ij}) \cdot A_{ij}$ is an

<u>estimate</u> for the partial volume between θ_j and θ_{j+1}, and

$$\sum_{j=1}^{m} \sum_{i=1}^{n} f(P_{ij}) A_{ij}$$

is an <u>estimate</u> for the total volume of the solid. The <u>exact</u> volume is

$$\int_R f(P)\,dA = \lim_{\text{mesh}\to 0} \sum_{j=1}^{m} \sum_{i=1}^{n} f(P_{ij})\,A_{ij}$$

$$= \lim_{\text{mesh}\to 0} \sum_{j=1}^{m} \left(\lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(r_i^*, \theta_j^*)\, r_i^* \Delta r_i \right) \Delta\theta_j$$

$$= \lim_{\text{mesh}\to 0} \sum_{j=1}^{m} \underbrace{\left(\int_a^b f(r, \theta_j^*)\, r\, dr \right)}_{g(\theta_j^*)} \Delta\theta_j$$

$$= \lim_{\text{mesh}\to 0} \sum_{j=1}^{m} g(\theta_j^*)\, \Delta\theta_j$$

$$= \int_\alpha^\beta g(\theta)\, d\theta$$

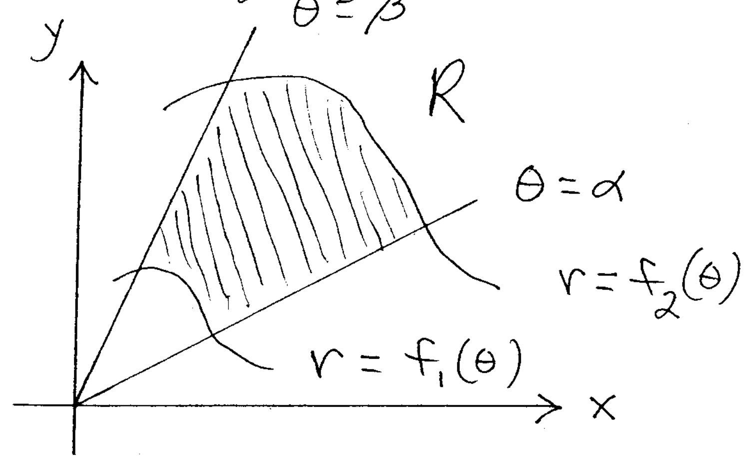
$$= \int_\alpha^\beta \left(\int_a^b f(r,\theta)\, r\, dr \right) d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta ,$$

i.e.,

$$\boxed{\int_R f(P)\,dA = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta}\ .$$

Similarly, but using more sophisticated means (e.g., partial derivatives, determinants, Jacobian, etc.), if region R is given in polar form by



$\alpha \le \theta \le \beta$

and

$f_1(\theta) \le r \le f_2(\theta)$,

then

$$\boxed{\int_R f(P)\, dA = \int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, r\, dr\, d\theta} \, .$$