

Math 21C

Kouba

The Ratio Test

Ratio Test: Assume $a_n > 0$ and consider $\sum_{n=1}^{\infty} a_n$.

1.) If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2.) If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3.) If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$, then the ratio test is inconclusive.

Proof: 1.) Assume $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1$. $\frac{L}{2} \quad \frac{L+1}{2} \quad 1$

Then there is some integer N so that

$\frac{a_{n+1}}{a_n} < \frac{L+1}{2} = m$ for all $n \geq N$. Thus,

$a_{n+1} < m a_n$ for all $n \geq N$, so that

$$a_{N+1} < m a_N,$$

$$a_{N+2} < m a_{N+1} < m^2 a_N,$$

$$a_{N+3} < m a_{N+2} < m^3 a_N,$$

$$a_{N+4} < m a_{N+3} < m^4 a_N, \dots$$

It follows that

$$\sum_{n=N}^{\infty} a_n = a_N + a_{N+1} + a_{N+2} + a_{N+3} + \dots$$

$$\leq a_N + m a_N + m^2 a_N + m^3 a_N + \dots$$

$$= a_N (1 + m + m^2 + m^3 + \dots)$$

$$= a_N \cdot \frac{1}{1-m} \quad (\text{since } -1 < m < 1)$$

$$< \infty.$$

This means that $\sum_{n=N}^{\infty} a_n$ converges, so that $\sum_{n=1}^{\infty} a_n$ converges.

2.) Assume $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L > 1$. Then there

is some integer N so that $\frac{a_{n+1}}{a_n} > 1$ for all $n \geq N$. Thus, $a_{n+1} > a_n$ for all $n \geq N$, so that

$$0 < a_N < a_{N+1} < a_{N+2} < a_{N+3} \dots$$

This means that a_n is a positive, increasing sequence for $n \geq N$. It follows that $\lim_{n \rightarrow \infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges by the n th term test.

QED