

Math 21C

Kouba

Problems Using (*) and (**)

$$\begin{aligned} 1.) \text{ a.) } \int_1^{11} \frac{1}{x} dx &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{10} < 1 + \int_1^{10} \frac{1}{x} dx \rightarrow \\ \ln x \Big|_1^{11} &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{10} < 1 + \ln x \Big|_1^{10} \rightarrow \\ \ln 11 - \ln 1 &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{10} < 1 + \ln 10 - \ln 1 \rightarrow \\ 2.398 \approx \ln 11 &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{10} < 1 + \ln 10 \approx 3.303 \end{aligned}$$

$$\begin{aligned} \text{b.) } \int_1^{1001} \frac{1}{x} dx &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1000} < 1 + \int_1^{1000} \frac{1}{x} dx \rightarrow \\ \int_1^{1001} \frac{1}{x} dx &= \ln x \Big|_1^{1001} = \ln 1001 - \ln 1 = \ln 1001 \text{ and} \\ \int_1^{1000} \frac{1}{x} dx &= \ln x \Big|_1^{1000} = \ln 1000 - \ln 1 = \ln 1000 \text{ so} \\ 6.909 \approx \ln 1001 &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1000} < 1 + \ln 1000 \approx 7.908 \end{aligned}$$

$$\begin{aligned} \text{c.) } \int_1^{1,000,001} \frac{1}{x} dx &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1,000,000} < 1 + \int_1^{1,000,000} \frac{1}{x} dx \rightarrow \\ 13.816 \approx \ln 1,000,001 &< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1,000,000} < 1 + \ln 1,000,000 \approx 14.816 \end{aligned}$$

$$2.) \text{ a.) } S_{10} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{10^2} \approx 1.550$$

$$\begin{aligned} \text{b.) } \int_{11}^{\infty} \frac{1}{x^2} dx &< \frac{1}{11^2} + \frac{1}{12^2} + \dots < \int_{10}^{\infty} \frac{1}{x^2} dx \rightarrow \\ \lim_{A \rightarrow \infty} \int_{11}^A \frac{1}{x^2} dx &= \lim_{A \rightarrow \infty} \left. \frac{-1}{x} \right|_{11}^A = \lim_{A \rightarrow \infty} \left(\frac{-1}{A} - \frac{-1}{11} \right) = \frac{1}{11} \text{ and} \\ \lim_{A \rightarrow \infty} \int_{10}^A \frac{1}{x^2} dx &= \lim_{A \rightarrow \infty} \left. \frac{-1}{x} \right|_{10}^A = \lim_{A \rightarrow \infty} \left(\frac{-1}{A} - \frac{-1}{10} \right) = \frac{1}{10} \text{ so} \end{aligned}$$

$$0.091 \approx \frac{1}{11} < \frac{1}{11^2} + \frac{1}{12^2} + \dots < \frac{1}{10} = 0.1$$

$$c.) \int_{101}^{\infty} \frac{1}{x^2} dx < R_{100} < \int_{100}^{\infty} \frac{1}{x^2} dx \rightarrow$$

$$\lim_{A \rightarrow \infty} \int_{101}^A \frac{1}{x^2} dx = \lim_{A \rightarrow \infty} \left. \frac{-1}{x} \right|_{101}^A = \lim_{A \rightarrow \infty} \left(\frac{-1}{A} - \frac{-1}{101} \right) = \frac{1}{101} \text{ and}$$

$$\lim_{A \rightarrow \infty} \int_{100}^A \frac{1}{x^2} dx = \lim_{A \rightarrow \infty} \left. \frac{-1}{x} \right|_{100}^A = \lim_{A \rightarrow \infty} \left(\frac{-1}{A} - \frac{-1}{100} \right) = \frac{1}{100} \text{ so}$$

$$0.0099 \approx \frac{1}{101} < R_{100} < \frac{1}{100} = 0.01$$

$$d.) R_n < \int_n^{\infty} \frac{1}{x^2} dx \leq 0.0001 \rightarrow$$

$$\lim_{A \rightarrow \infty} \int_n^A \frac{1}{x^2} dx \leq 0.0001 \rightarrow \lim_{A \rightarrow \infty} \left. \frac{-1}{x} \right|_n^A \leq 0.0001$$

$$\rightarrow \lim_{A \rightarrow \infty} \left(\frac{-1}{A} - \frac{-1}{n} \right) \leq 0.0001 \rightarrow \frac{1}{n} \leq 0.0001 \rightarrow$$

$$n \geq 10,000$$

$$3.) a.) S_{10} = 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^9 = \frac{1 - \left(\frac{2}{3}\right)^{10}}{1 - \left(\frac{2}{3}\right)} \approx 2.9480$$

$$b.) \int_{11}^{\infty} \left(\frac{2}{3}\right)^{x-1} dx < R_{10} < \int_{10}^{\infty} \left(\frac{2}{3}\right)^{x-1} dx \rightarrow$$

$$\lim_{A \rightarrow \infty} \int_{11}^A \left(\frac{2}{3}\right)^{x-1} dx = \lim_{A \rightarrow \infty} \left. \frac{\left(\frac{2}{3}\right)^{x-1}}{\ln\left(\frac{2}{3}\right)} \right|_{11}^A$$

$$= \lim_{A \rightarrow \infty} \left(\frac{\left(\frac{2}{3}\right)^{A-1}}{\ln\left(\frac{2}{3}\right)} - \frac{\left(\frac{2}{3}\right)^{10}}{\ln\left(\frac{2}{3}\right)} \right) = \frac{-\left(\frac{2}{3}\right)^{10}}{\ln\left(\frac{2}{3}\right)} \approx 0.0427 \text{ and}$$

$$\lim_{A \rightarrow \infty} \int_{10}^A \left(\frac{2}{3}\right)^{x-1} dx = \lim_{A \rightarrow \infty} \left. \frac{\left(\frac{2}{3}\right)^{x-1}}{\ln\left(\frac{2}{3}\right)} \right|_{10}^A$$

$$= \lim_{A \rightarrow \infty} \left(\frac{\left(\frac{2}{3}\right)^{A-1}}{\ln(2/3)} - \frac{\left(\frac{2}{3}\right)^9}{\ln(2/3)} \right) = \frac{-\left(\frac{2}{3}\right)^9}{\ln(2/3)} \approx 0.0642 \text{ so}$$

$$0.0427 < R_{10} < 0.0642 .$$

$$\begin{aligned} \text{c.) } R_{10} &= \left(\frac{2}{3}\right)^{10} + \left(\frac{2}{3}\right)^{11} + \left(\frac{2}{3}\right)^{12} + \dots = \left(\frac{2}{3}\right)^{10} \cdot \left[1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \dots\right] \\ &= \left(\frac{2}{3}\right)^{10} \cdot \frac{1}{1 - \left(\frac{2}{3}\right)} \approx 0.0520 \end{aligned}$$

$$\text{d.) } R_n < \int_n^{\infty} \left(\frac{2}{3}\right)^{x-1} dx \leq 0.0001 \rightarrow$$

$$\lim_{A \rightarrow \infty} \int_n^A \left(\frac{2}{3}\right)^{x-1} dx \leq 0.0001 \rightarrow$$

$$\lim_{A \rightarrow \infty} \left. \frac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)} \right|_n^A = \lim_{A \rightarrow \infty} \left(\frac{\left(\frac{2}{3}\right)^{A-1}}{\ln(2/3)} - \frac{\left(\frac{2}{3}\right)^{n-1}}{\ln(2/3)} \right) \leq 0.0001 \rightarrow$$

$$\frac{-\left(\frac{2}{3}\right)^{n-1}}{\ln(2/3)} \leq 0.0001 \rightarrow \left(\frac{2}{3}\right)^{n-1} \leq -(0.0001) \ln(2/3) \rightarrow$$

$$(n-1) \ln\left(\frac{2}{3}\right) \leq \ln[(0.0001) \ln(2/3)] \rightarrow$$

$$n-1 \geq \frac{\ln[(0.0001) \ln(2/3)]}{\ln(2/3)} \rightarrow$$

$$n \geq \frac{\ln[(0.0001) \ln(2/3)]}{\ln(2/3)} + 1 \approx 25.9 \text{ so}$$

choose $n \geq 26$.

$$\begin{aligned} \text{e.) } \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} &= 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots \\ &= \frac{1}{1 - \left(\frac{2}{3}\right)} = 3 . \end{aligned}$$