Math 21C
Kouba - Integral Test

Assume that $f$ is a positive, decreasing, continuous function for $x \geq 1$.

\[ \int_{1}^{n+1} f(x) \, dx < f(1) + f(2) + f(3) + \cdots + f(n) \]

\[ f(2) + f(3) + \cdots + f(n) < \int_{1}^{n} f(x) \, dx \Rightarrow f(1) + f(2) + f(3) + \cdots + f(n) < f(1) + \int_{1}^{n} f(x) \, dx \]
It follows that

\[
\sum_{i=1}^{n+1} f(x) \, dx < f(1) + f(2) + \cdots + f(n) < f(1) + \sum_{i=1}^{n} f(x) \, dx.
\]

\[\text{(\ast)}\]

**Case 1:** Assume that \( \int_{1}^{\infty} f(x) \, dx = L \) (finite). Then

\[
\lim_{n \to \infty} \left( f(1) + f(2) + \cdots + f(n) \right) < \lim_{n \to \infty} \left( f(1) + \int_{1}^{n} f(x) \, dx \right) \Rightarrow
\]

\[
\sum_{n=1}^{\infty} f(n) < f(1) + \int_{1}^{\infty} f(x) \, dx = f(1) + L < \infty, \text{ i.e.,}
\]

\[
\sum_{n=1}^{\infty} f(n) \text{ converges}.
\]

**Case 2:** Assume that \( \int_{1}^{\infty} f(x) \, dx = \infty \). Then

\[
\lim_{n \to \infty} \int_{1}^{n+1} f(x) \, dx \leq \lim_{n \to \infty} \left( f(1) + f(2) + \cdots + f(n) \right) \Rightarrow
\]

\[
\int_{1}^{\infty} f(x) \, dx \leq \sum_{n=1}^{\infty} f(n) \Rightarrow \sum_{n=1}^{\infty} f(n) = \infty, \text{ i.e.,}
\]

\[
\sum_{n=1}^{\infty} f(n) \text{ diverges}.
\]

This verifies the following series test.

**Integral Test:** Assume that function \( f \) is positive, decreasing, and continuous for \( x \geq 1 \).

a.) If \( \int_{1}^{\infty} f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} f(n) \) converges.

b.) If \( \int_{1}^{\infty} f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} f(n) \) diverges.
Note that
\[ \sum_{n=1}^{\infty} f(n) = \underbrace{f(1) + f(2) + \cdots + f(n)}_{S_n} + \underbrace{f(n+1) + f(n+2) + \cdots}_{R_n} \]

Let \( S_n = f(1) + f(2) + \cdots + f(n) \) be the \( n \)th partial sum, and let the infinite tail
\[ R_n = f(n+1) + f(n+2) + \cdots \]

be the error or remainder term for the infinite series \( \sum_{n=1}^{\infty} f(n) \). Then

\[ y = f(x) \]

\[ f(n+1) + f(n+2) + f(n+3) + \cdots < \int_{n}^{\infty} f(x) \, dx, \]

and

\[ \int_{n+1}^{\infty} f(x) \, dx < f(n+1) + f(n+2) + \cdots, \text{ so that} \]

\[ \int_{n+1}^{\infty} f(x) \, dx < f(n+1) + f(n+2) + \cdots < \int_{n}^{\infty} f(x) \, dx \]