Math 21C
Kouba
Triple Integrals Over Solid Regions $R$ in Three-Dimensional Space
Using Rectangular Coordinates

Consider a solid region $R$ in three-dimensional space and let $w = f(P)$ be a function of three variables defined at each point $P = (x, y, z)$ in $R$. First partition the solid region $R$ into $n$ parts $R_1, R_2, R_3,$ and $R_n$ of volumes $V_1, V_2, V_3,$ and $V_n$, resp. Pick sampling point $P_i = (x_i, y_i, z_i)$ in region $R_i$ for $i = 1, 2, 3, \cdots, n$. Define the diameter of solid region $R_i$, $diam(R_i)$, to be the maximum distance between points in $R_i$ for $i = 1, 2, 3, \cdots, n$. Define the mesh of the partition to be

$$\text{mesh} = \max_{1 \leq i \leq n} (diam(R_i)).$$

Now we define the integral of $f$ over the solid region $R$ to be

$$\int_R f(P) \, dV = \lim_{\text{mesh} \to 0} \sum_{i=1}^n f(P_i) \cdot V_i.$$

In order to motivate the actual evaluation of this integral assume that function $w = f(P)$ represents the density (mass/volume units) at point $P = (x, y, z)$ in $R$. Then $\int_R f(P) \, dV$ represents the total mass of solid region $R$. Assume that solid region $R$ is described by

$$a \leq x \leq b, \quad g(x) \leq y \leq k(x), \quad \text{and} \quad u(x, y) \leq z \leq v(x, y).$$
Next let \( a = x_0, x_1, x_2, x_3, \) and \( x_n = b \) partition the interval \([a, b]\) into \( n \) parts. Pick sampling point \( x_i \) and let \( \Delta x_i = x_i - x_{i-1} \) for \( i = 1, 2, 3, \ldots, n \). Define the mesh of the partition to be \( \text{mesh} = \max_{1 \leq i \leq n} (x_i - x_{i-1}) \). Make a slice through the solid region \( R \) perpendicular to the \( x \)-axis at point \( x_i \) and let \( R(x_i) \) represent the flat intersection of this plane with the solid region \( R \). \( R(x_i) \) can be described by

\[
g(x_i) \leq y \leq k(x_i) \quad \text{and} \quad u(x_i, y) \leq z \leq v(x_i, y)
\]

It follows that

\[
\int_{g(x_i)}^{k(x_i)} \int_{u(x_i, y)}^{v(x_i, y)} f(x_i, y, z) \, dz \, dy
\]

is a measure in \((\text{mass/volume})(\text{area}) = (\text{mass/length}) \) units,

\[
\left( \int_{g(x_i)}^{k(x_i)} \int_{u(x_i, y)}^{v(x_i, y)} f(x_i, y, z) \, dz \, dy \right) \Delta x_i
\]

is an estimate for the mass of the slice \( R(x_i) \) of thickness \( \Delta x_i \) with units \((\text{mass})\), and

\[
\lim_{\text{mesh} \to 0} \sum_{i=1}^{n} \left( \int_{g(x_i)}^{k(x_i)} \int_{u(x_i, y)}^{v(x_i, y)} f(x_i, y, z) \, dz \, dy \right) \Delta x_i = \int_{a}^{b} \int_{g(x)}^{k(x)} \int_{u(x, y)}^{v(x, y)} f(x, y, z) \, dz \, dy \, dx
\]

We can now conclude that

\[
\int_{R} f(P) \, dV = \int_{a}^{b} \int_{g(x)}^{k(x)} \int_{u(x, y)}^{v(x, y)} f(x, y, z) \, dz \, dy \, dx
\]