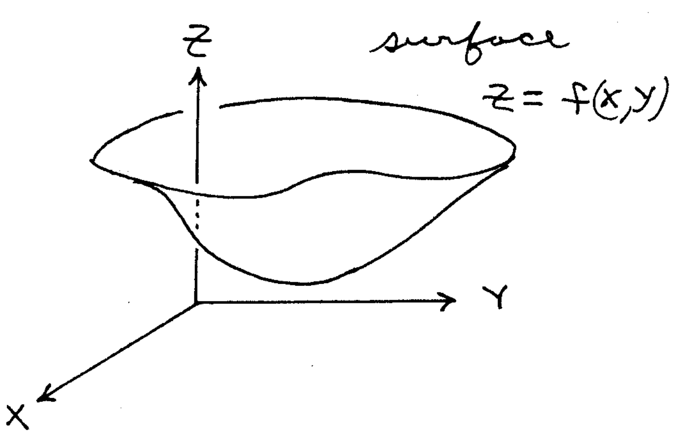
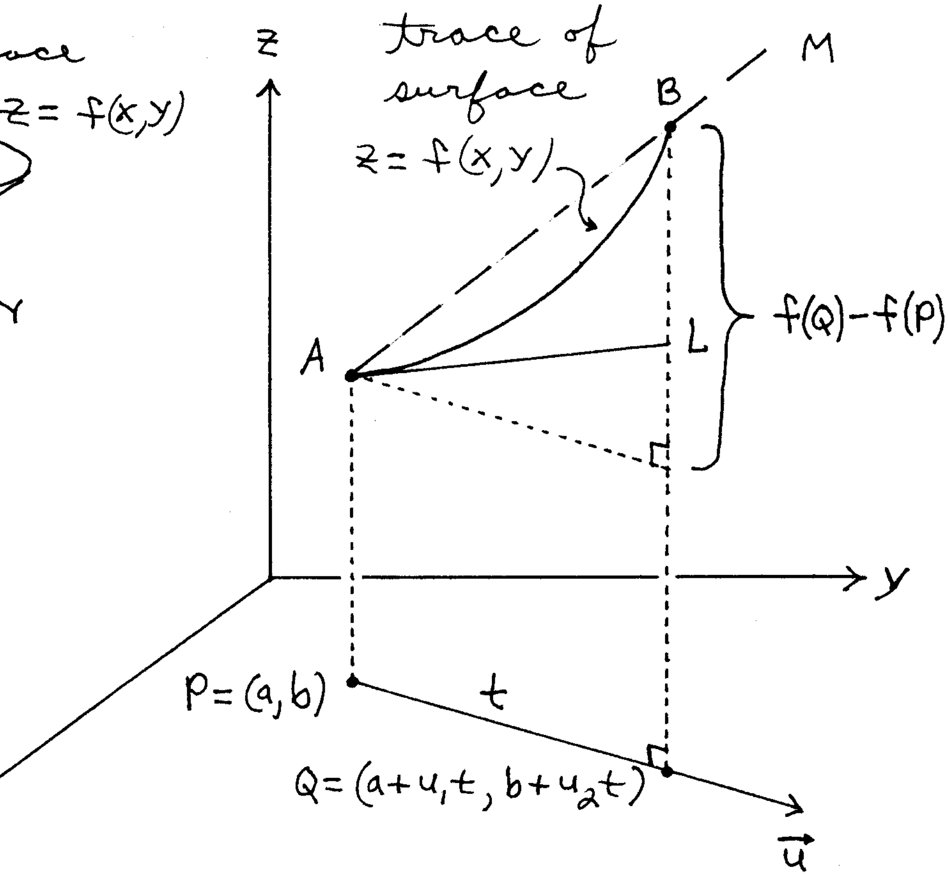
Math 21C
Koruba
Directional Derivatives and
Gradient Vectors

Consider a surface given by the function $z = f(x,y)$, a point $P = (a,b)$, and a unit vector $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$. We seek to define the derivative of f at point P in the direction of $\vec{u}$:





Remarks :

1.) Let point $Q = (a + u_1 t,\ b + u_2 t)$, where $t \geq 0$.

2.) Vector $\overrightarrow{PQ} = \overrightarrow{(u_1 t,\ u_2 t)} = t\,\overrightarrow{(u_1, u_2)} = t\,\vec{u}$

points in the direction of $\vec{u}$ and has length t since $\vec{u}$ is a unit vector.

3.) Line M is the secant line through points A and B .

4.) Line L is the line tangent to the trace of the surface $z = f(x, y)$ at the point $A = (a, b, f(a,b))$ in the direction of vector $\vec{u}$.

5.) The slope of line M is

$$\frac{\text{rise}}{\text{run}} = \frac{f(Q) - f(P)}{t} = \frac{f(a + u_1 t, b + u_2 t) - f(a,b)}{t} .$$

6.) The slope of line L is

$$\lim_{t \to 0} \frac{f(a + u_1 t, b + u_2 t) - f(a,b)}{t} .$$

Def: Consider function $z = f(x, y)$, point $P = (a, b)$, and unit vector $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$. The derivative of f at point P in the direction $\vec{u}$ is

$$(\mathrm{I}) \qquad D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{f(a + u_1 t, b + u_2 t) - f(a,b)}{t} .$$

Remark : $D_{\vec{u}} f(a,b)$ is the slope of tangent line L .

Now consider the function

$g(t) = f(a + u_1 t, b + u_2 t)$, or also written

$g(t) = f(x,y)$, where $x = a + u_1 t$ and $y = b + u_2 t$.

$$\begin{tikzcd}[column sep=small] & g=f \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & t & \end{tikzcd}$$

By the chain rule it follows that

$$g'(t) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \overrightarrow{\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)} \bullet \overrightarrow{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}$$

$$= \overrightarrow{\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)} \bullet \overrightarrow{(u_1, u_2)}$$

$$= \overrightarrow{\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)} \bullet \vec{u}\ ,$$ so that

(II) $$g'(0) = \overrightarrow{\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)} \bullet \vec{u} \quad .$$

Note also that $g(0) = f(a,b)$ so from (I) we get that

$$D_{\vec{u}}\, f(a,b) = \lim_{t\to 0} \frac{g(t) - g(0)}{t - 0} = g'(0) \ .$$

It now follows from (II) that

(III) $$D_{\vec{u}}\, f(a,b) = \overrightarrow{\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)} \bullet \vec{u} \quad .$$

<u>Def</u>: The <u>gradient</u> <u>vector</u> of $z = f(x,y)$ at point $P = (a,b)$ is given by

$$\vec{\nabla} f = \frac{\partial f}{\partial x}(a,b)\,\vec{i} + \frac{\partial f}{\partial y}(a,b)\,\vec{j} \quad .$$

<u>Notation</u>: $D_{\vec{u}}\, f(a,b) = \vec{\nabla} f \bullet \vec{u}$.