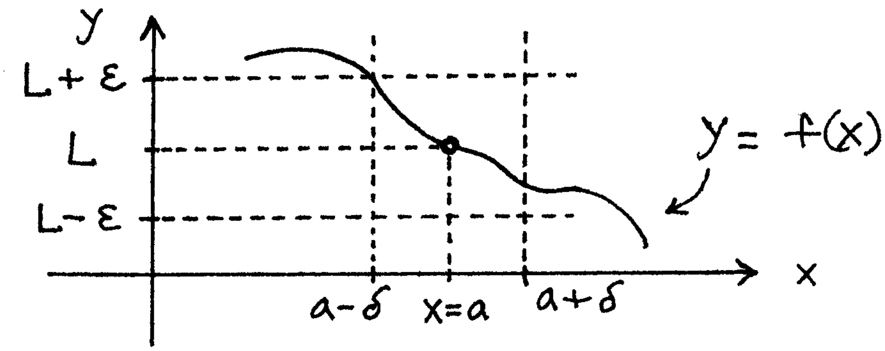
Math 21C
Kouba
Limits of Functions of Two Variables

RECALL (from Math 21A) : $\lim_{x \to a} f(x) = L$ means : For each $\epsilon > 0$ there exists a $\delta > 0$ so that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.



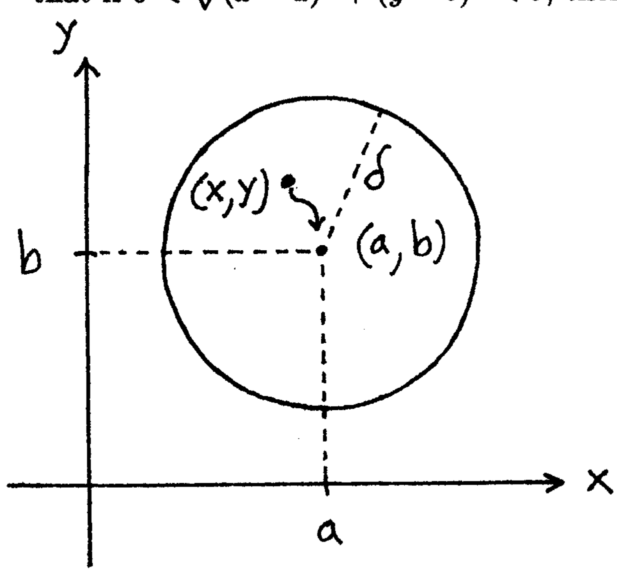
RECALL (from Math 21A) : Function f is continuous at $x = a$ if

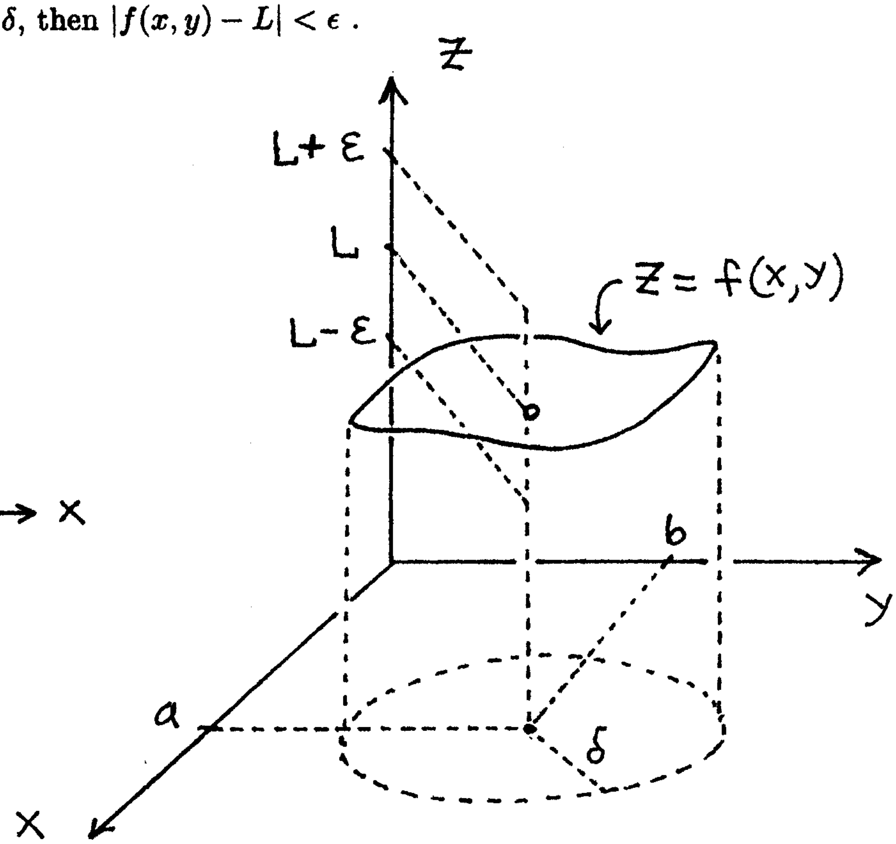
1.) $f(a)$ is defined (finite),
2.) $\lim_{x \to a} f(x) = L$ (finite),

and

3.) $\lim_{x \to a} f(x) = f(a)$.

DEFINITION : $\lim_{(x,y) \to (a,b)} f(x,y) = L$ means : For each $\epsilon > 0$ there exists a $\delta > 0$ so that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$.





Ex: Prove that $\lim\limits_{(x,y)\to(1,-1)} (x^2-y) = 2$:

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-1)^2+(y-(-1))^2} = \sqrt{(x-1)^2+(y+1)^2} < \delta$,

then $|(x^2-y)-2| < \varepsilon$. Then

$$|x^2-y-2| = |(x-1)^2+2x-\cancel{1}-(y+1)+\cancel{1}-2|$$

$$= |(x-1)^2+2(x-1)-(y+1)|$$

Δ-inequality

$$\le |(x-1)^2| + |2(x-1)| + |y+1|$$

$$= (x-1)^2 + 2|x-1| + |y+1|$$

$$= (x-1)^2 + 2\sqrt{(x-1)^2} + \sqrt{(y+1)^2}$$

$$\le (x-1)^2+(y+1)^2+2\sqrt{(x-1)^2+(y+1)^2}+\sqrt{(x-1)^2+(y+1)^2}$$

$$= \left(\sqrt{(x-1)^2+(y+1)^2}\right)^2 + 3\sqrt{(x-1)^2+(y+1)^2}$$

assume $\delta \le 1$ so that $A^2 \le A$

$$\le \sqrt{(x-1)^2+(y+1)^2} + 3\sqrt{(x-1)^2+(y+1)^2}$$

$$= 4\sqrt{(x-1)^2+(y+1)^2} < \varepsilon$$

iff $\sqrt{(x-1)^2+(y+1)^2} < \frac{1}{4}\varepsilon$.

Now choose $\delta = \min\{\frac{1}{4}\varepsilon, 1\}$ and the result follows.

QED