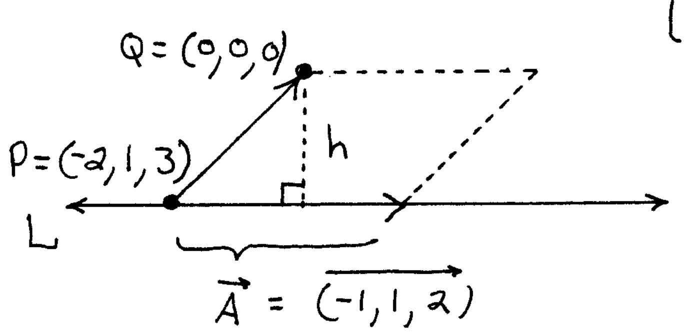
Math 21C
Kouba
Examples Using Dot Products, Cross Products, and Projections

Example 1: Find the distance from the point $(0,0,0)$ to line $L: \begin{cases} x = -2 - t \\ y = 1 + t \\ z = 3 + 2t \end{cases}$: Note that



pt. $P = (-2,1,3)$ is on L and vector $\vec{A} = \overrightarrow{(-1,1,2)}$ is $\parallel$ to L. The area of the parallelogram formed by vectors $\overrightarrow{PQ}$ and $\vec{A}$

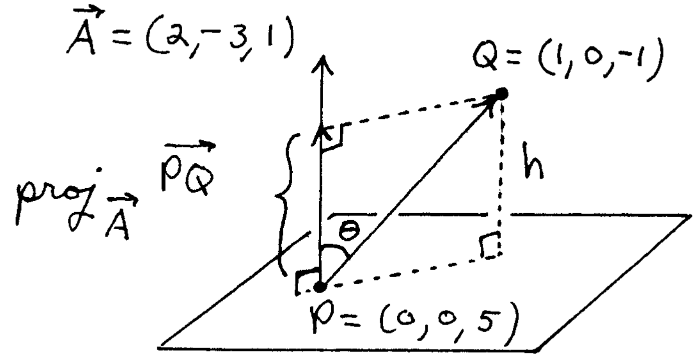
is $|\overrightarrow{PQ} \times \vec{A}| = (\text{base})(\text{height}) = |\vec{A}| \cdot (h) \Rightarrow$

$$h = \frac{|\overrightarrow{PQ} \times \vec{A}|}{|\vec{A}|} = \frac{|\overrightarrow{(2,-1,-3)} \times \overrightarrow{(-1,1,2)}|}{\sqrt{6}} = \frac{|\overrightarrow{(1,-1,1)}|}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

$$\overrightarrow{PQ} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{vmatrix} = (-2+3)\vec{i} - (4-3)\vec{j} + (2-1)\vec{k} = \vec{i} - \vec{j} + \vec{k}$$

Example 2: Find the distance from the point $(1,0,-1)$ to the plane $2x - 3y + z = 5$:

Note that pt. $P = (0,0,5)$ lies on the plane and vector $\vec{A} = \overrightarrow{(2,-3,1)}$ is $\perp$ to the plane.



Then distance h is given by

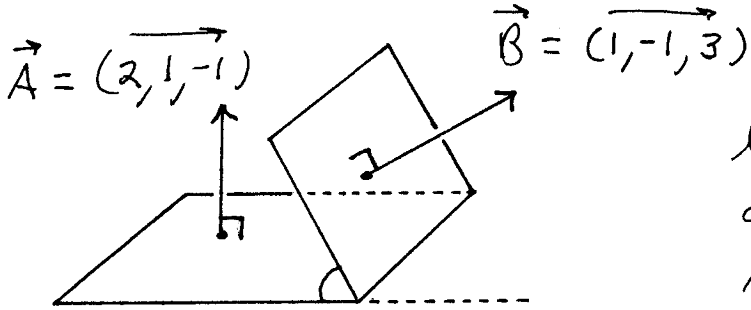
$$h = |\text{proj}_{\vec{A}}\ \overrightarrow{PQ}|$$

$$= |\overrightarrow{PQ}|\,|\cos\theta|$$

$$= |\overrightarrow{PQ}| \left| \frac{\vec{A}\cdot\overrightarrow{PQ}}{|\vec{A}|\,|\overrightarrow{PQ}|} \right|$$

$$= \frac{|\vec{A}\cdot\overrightarrow{PQ}|}{|\vec{A}|} = \frac{|\overrightarrow{(2,-3,1)}\cdot\overrightarrow{(1,0,-6)}|}{|\overrightarrow{(2,-3,1)}|} = \frac{|2+0-6|}{\sqrt{14}} = \frac{4}{\sqrt{14}}$$

<u>Example 3</u>: Find the angle θ between the planes $2X+Y-Z=0$ and $X-Y+3Z=1$:



The angle θ can be determined by the angle α between the vectors $\perp$ to each plane : $\vec{A} = \overrightarrow{(2,1,-1)}$ and $\vec{B} = \overrightarrow{(1,-1,3)}$.

The angle α between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\alpha = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{2-1-3}{\sqrt{6}\,\sqrt{11}} = \frac{-2}{\sqrt{66}} \Rightarrow$$

$$\alpha = \arccos\left(\frac{-2}{\sqrt{66}}\right) \approx 104.3^\circ \Rightarrow \text{angle}$$

between planes is $\theta \approx 180^\circ - \alpha = 75.7^\circ$