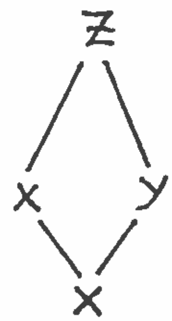
Math 21C
Kouba
Gradient Vectors and Vectors Normal to Level Curves

Partial Derivatives and Implicit Differentiation : Assume that function $F(x, y) = c$, where c is a constant and $y = g(x)$, is an equation in x and y. We will show here a new way to find the ordinary derivative $y' = \dfrac{dy}{dx}$ using the Chain Rule for partial derivatives. From the diagram and the Chain Rule we get



$$\frac{dF(x,y)}{dx} = \frac{dc}{dx} \longrightarrow$$

$$\frac{dF}{dx} = F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} = 0 \longrightarrow$$

$$F_x \cdot \{1\} + F_y \cdot \frac{dy}{dx} = 0 \longrightarrow \qquad (\#)\ \frac{dy}{dx} = \frac{-F_x}{F_y}$$

DEFINITION : (I.) Let $z = f(x, y)$ be a function. The *gradient vector* of f at the point (x, y) is

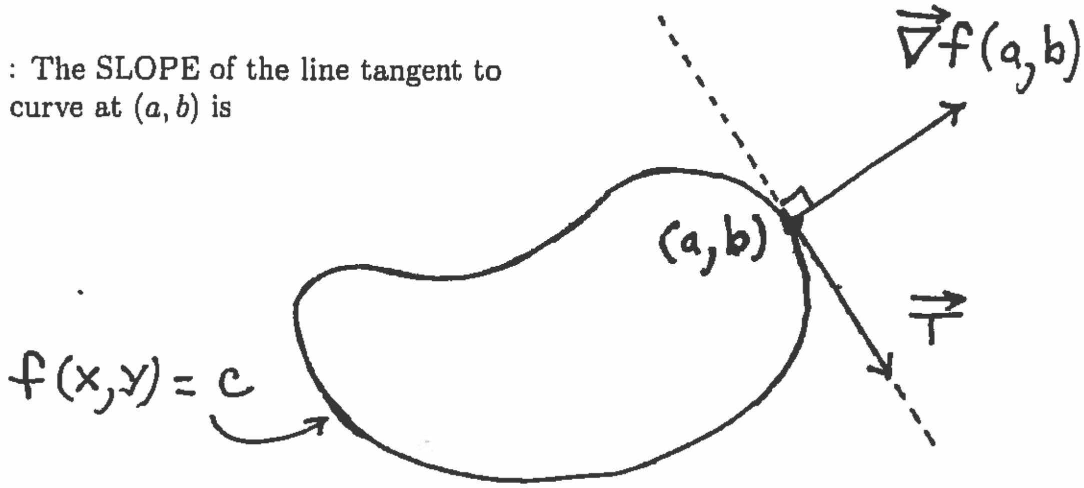
$$\vec{\nabla} f = \overrightarrow{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

(II.) Let $w = f(x, y, z)$ be a function. The *gradient vector* of f at the point (x, y, z) is

$$\vec{\nabla} f = \overrightarrow{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}$$

THEOREM (Gradient Vectors and Level Curves) : (I.) Let $z = f(x, y)$ be a function and let $(x, y) = (a, b)$ with $f(a, b) = c$. Then the gradient vector $\vec{\nabla} f(a, b)$ is normal (perpendicular) to the *level curve* $f(x, y) = c$.

PROOF : The SLOPE of the line tangent to the level curve at (a, b) is



$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x}$$

so that a vector parallel to the tangent line at (a, b) is

$$\vec{T} = \overrightarrow{(f_y, -f_x)} .$$

(Recall that vectors $\vec{v}$ and $\vec{w}$ are orthogonal (perpendicular) iff $\vec{v} \bullet \vec{w} = 0$.) Then

$$\vec{T} \bullet \vec{\nabla} f(a, b) = \overrightarrow{(f_y, -f_x)} \bullet \overrightarrow{(f_x, f_y)} = f_x f_y - f_x f_y = 0 .$$ QED

THEOREM (Gradient Vectors and Level Curves) : (II.) Let $w = f(x, y, z)$ be a function and let $(x, y, z) = (a, b, c)$ with $f(a, b, c) = d$. Then the gradient vector $\vec{\nabla} f(a, b, c)$ is normal (perpendicular) to the *level curve* $f(x, y, z) = d$.

PROOF : This requires the concept of a vector function, which will be coverd in Math 21D.