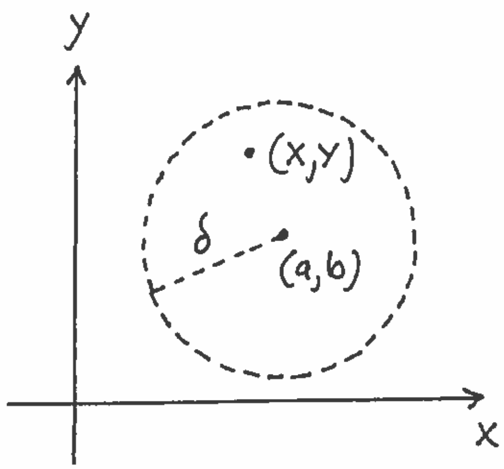
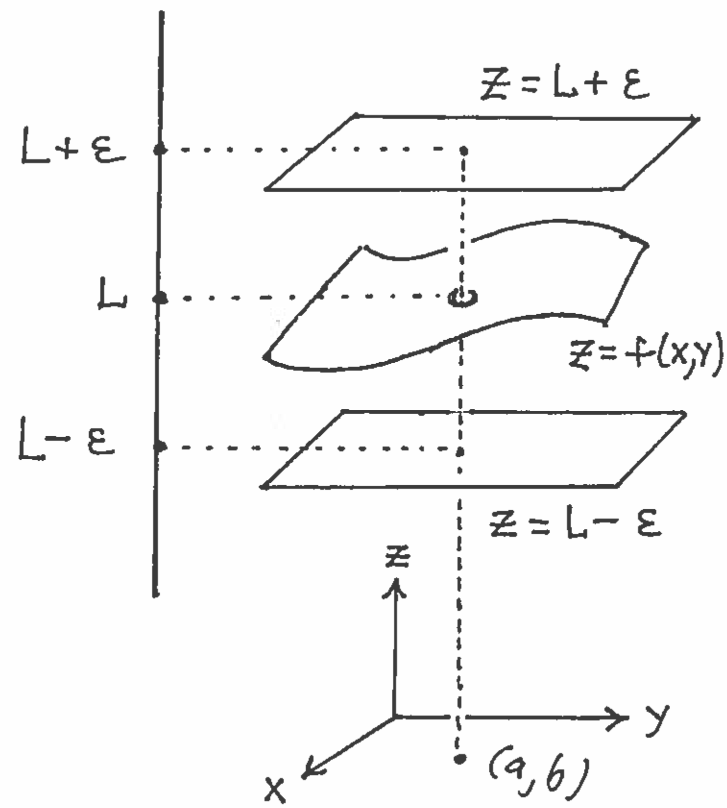
Math 21C
Kouba
Precise Limits and Continuity for Functions of Two Variables

DEFINITION : Let $z = f(x, y)$ be a function. The limit $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$ means that for each $\epsilon > 0$ there exist a $\delta > 0$ so that

$$\text{if} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \ , \quad \text{then} \quad |f(x, y) - L| < \epsilon \ .$$





DEFINITION : Function $z = f(x, y)$ is continuous at point (a, b) iff

i.) $f(a, b)$ is defined (finite) ,

ii.) $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$ (finite) ,

and iii.) $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = L$.

Ex: Prove that $\displaystyle\lim_{(x,y)\to(1,-1)} (x^2-y) = 2$:

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-1)^2+(y-(-1))^2} = \sqrt{(x-1)^2+(y+1)^2} < \delta$,

then $|(x^2-y)-2| < \varepsilon$. Then

$$|x^2-y-2| = |(x-1)^2+2x-\cancel{1}-(y+1)+\cancel{1}-2|$$

$$= |(x-1)^2+2(x-1)-(y+1)|$$

$$\underset{\Delta\text{ inequality}}{\le} |(x-1)^2| + |2(x-1)| + |y+1|$$

$$= (x-1)^2 + 2|x-1| + |y+1|$$

$$= (x-1)^2 + 2\sqrt{(x-1)^2} + \sqrt{(y+1)^2}$$

$$\le (x-1)^2+(y+1)^2 + 2\sqrt{(x-1)^2+(y+1)^2} + \sqrt{(x-1)^2+(y+1)^2}$$

$$= \left(\sqrt{(x-1)^2+(y+1)^2}\right)^2 + 3\sqrt{(x-1)^2+(y+1)^2}$$

assume $\delta \le 1$ so that $A^2 \le A$

$$\le \sqrt{(x-1)^2+(y+1)^2} + 3\sqrt{(x-1)^2+(y+1)^2}$$

$$= 4\sqrt{(x-1)^2+(y+1)^2} < \varepsilon$$

iff $\sqrt{(x-1)^2+(y+1)^2} < \frac{1}{4}\varepsilon$.

Now choose

$\delta = \min\{\frac{1}{4}\varepsilon, 1\}$ and the result follows.

QED