DEFINITION: Let \( z = f(x, y) \) be a function. The limit \( \lim_{(x,y) \to (a,b)} f(x, y) = L \) means that for each \( \epsilon > 0 \) there exist a \( \delta > 0 \) so that

\[
\text{if } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta, \text{ then } |f(x, y) - L| < \epsilon.
\]

DEFINITION: Function \( z = f(x, y) \) is continuous at point \((a, b)\) iff

i.) \( f(a, b) \) is defined (finite),

ii.) \( \lim_{(x,y) \to (a,b)} f(x, y) = L \) (finite),

and iii.) \( \lim_{(x,y) \to (a,b)} f(x, y) = L \).
Ex: Prove that \( \lim_{(x,y) \to (1,1)} (x^2 - y) = 2 \).

Let \( \varepsilon > 0 \) be given. Find \( \delta > 0 \) so that

\[
0 < \sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-1)^2 + (y+1)^2} < \delta,
\]

then \( |(x^2 - y) - 2| < \varepsilon \). Then

\[
|x^2 - y - 2| = |(x-1)^2 + 2x - 1 - (y+1) + 1 - 2|
\]

\[
= |(x-1)^2 + 2(x-1) - (y+1)|
\]

By inequality,

\[
|a - b| \leq |a|^2 + |b|^2 + |a + b|
\]

\[
= (x-1)^2 + 2|x-1| + |y+1|
\]

\[
= (x-1)^2 + 2\sqrt{(x-1)^2 + (y+1)^2}
\]

\[
\leq (x-1)^2 + (y+1)^2 + 2\sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x-1)^2 + (y+1)^2}
\]

\[
= (\sqrt{(x-1)^2 + (y+1)^2})^2 + 3\sqrt{(x-1)^2 + (y+1)^2}
\]

Assume \( \delta \leq 1 \) so that

\[
\sqrt{(x-1)^2 + (y+1)^2} < \frac{1}{4} \varepsilon
\]

if \( \sqrt{(x-1)^2 + (y+1)^2} < \frac{1}{4} \varepsilon \).

Now choose

\[
\delta = \min \left\{ \frac{1}{4} \varepsilon, 1 \right\}
\]

and the result follows. QED