Worksheet 2

1.) a.) $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$,

then $\left|\sqrt{x^2+y^2} - 0\right| = \sqrt{x^2+y^2} < \varepsilon$.

Choose $\delta = \varepsilon$ and the result follows.

QED

b.) $\lim\limits_{(x,y)\to(0,0)} (2x^2+3y^2) = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$,

then $\left|(2x^2+3y^2) - 0\right| = \left|2x^2+3y^2\right| < \varepsilon$. Then

$$\begin{aligned}
\left|2x^2+3y^2\right| &= 2x^2+3y^2 \\
&\le 3x^2+3y^2 \\
&= 3(x^2+y^2) \\
&= 3\left(\sqrt{x^2+y^2}\right)^2 < \varepsilon
\end{aligned}$$

iff $\left(\sqrt{x^2+y^2}\right)^2 < \frac{1}{3}\varepsilon$

iff $\sqrt{x^2+y^2} < \sqrt{\frac{1}{3}\varepsilon}$. Now

choose $\delta = \sqrt{\frac{1}{3}\varepsilon}$ and the result follows.

QED

c.) $\lim_{(x,y)\to(1,-1)} (x+y) = 0$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-1)^2 + (y-(-1))^2} = \sqrt{(x-1)^2+(y+1)^2} < \delta$,

then $|(x+y) - 0| = |x+y| < \varepsilon$. Then

$$|x+y| = |(x-1)+(y+1)|$$

$$\overset{\Delta\text{-inequality}}{\leq} |x-1| + |y+1|$$

$$= \sqrt{(x-1)^2} + \sqrt{(y+1)^2}$$

$$\leq \sqrt{(x-1)^2+(y+1)^2} + \sqrt{(x-1)^2+(y+1)^2}$$

$$= 2\sqrt{(x-1)^2+(y+1)^2} < \varepsilon$$

iff $\sqrt{(x-1)^2+(y+1)^2} < \frac{1}{2}\varepsilon$. Now choose $\delta = \varepsilon/2$ and the result follows. QED

d.) $\lim_{(x,y)\to(0,2)} (2x-y+1) = -1$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

if $0 < \sqrt{(x-0)^2+(y-2)^2} = \sqrt{x^2+(y-2)^2} < \delta$, then

$$|(2x-y+1) - (-1)| = |2x-y+2|$$

$$= |2(x) - (y-2) + 2 - 2|$$

$$= |2x - (y-2)|$$

$$\overset{\Delta\text{-inequality}}{\leq} |2x| + |y-2|$$

$$= 2|x| + |y-2|$$

$$= 2\sqrt{x^2} + \sqrt{(y-2)^2}$$

$$\leq 2\sqrt{x^2+(y-2)^2} + \sqrt{x^2+(y-2)^2}$$

$$= 3\sqrt{x^2+(y-2)^2} < \varepsilon$$

iff $\sqrt{x^2+(y-2)^2} < \frac{1}{3}\varepsilon$. Now choose $\delta = \frac{1}{3}\varepsilon$ and the result follows. QED

e.) $\lim\limits_{(x,y)\to(3,2)} xy = 6$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x-3)^2+(y-2)^2} < \delta$, then $|xy-6| < \varepsilon$.

Then $|xy-6| = |(x-3)(y-2) - 6 + 2x + 3y - 6|$

$$= |(x-3)(y-2) + 2x + 3y - 12|$$

$$= |(x-3)(y-2) + 2(x-3) + 3(y-2) - 12 + 6 + 6|$$

$$= |(x-3)(y-2) + 2(x-3) + 3(y-2)|$$

Δ-inequality

$$\leq |(x-3)(y-2)| + |2(x-3)| + |3(y-2)|$$

$$= |x-3||y-2| + 2|x-3| + 3|y-2|$$

$$= \sqrt{(x-3)^2}\cdot\sqrt{(y-2)^2} + 2\sqrt{(x-3)^2} + 3\sqrt{(y-2)^2}$$

$$\leq \sqrt{(x-3)^2+(y-2)^2}\cdot\sqrt{(x-3)^2+(y-2)^2} + 2\sqrt{(x-3)^2+(y-2)^2} + 3\sqrt{(x-3)^2+(y-2)^2}$$

$$= \left(\sqrt{(x-3)^2+(y-2)^2}\right)^2 + 5\sqrt{(x-3)^2+(y-2)^2}$$

assume $\delta \le 1$ so $A^2 \le A$

$$\le \sqrt{(x-3)^2+(y-2)^2} + 5\sqrt{(x-3)^2+(y-2)^2}$$

$$= 6\sqrt{(x-3)^2+(y-2)^2} < \varepsilon$$

iff $\sqrt{(x-3)^2+(y-2)^2} < \frac{1}{6}\varepsilon$. Now choose $\delta = \min\{\frac{1}{6}\varepsilon, 1\}$ and the result follows.

QED

f.) $\lim\limits_{(x,y)\to(-2,-1)} (3xy - y^2) = 5$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x+2)^2+(y+1)^2} < \delta$, then $|(3xy-y^2)-5| < \varepsilon$.

Then $|3xy - y^2 - 5| = |3(x+2)y - 6y - y^2 - 5|$

$$= |3(x+2)(y+1) - 3(x+2) - 6(y+1) + 6 - y^2 - 5|$$

$$= |3(x+2)(y+1) - 3(x+2) - 6(y+1) + 1 - (y+1)^2 + 2y + 1|$$

$$= |3(x+2)(y+1) - 3(x+2) - 6(y+1) - (y+1)^2 + 2(y+1)|$$

$$= |3(x+2)(y+1) - 3(x+2) - 4(y+1) - (y+1)^2|$$

Δ-inequality

$$\le 3|x+2|\,|y+1| + 3|x+2| + 4|y+1| + (y+1)^2$$

$$= 3\sqrt{(x+2)^2}\cdot\sqrt{(y+1)^2} + 3\sqrt{(x+2)^2} + 4\sqrt{(y+1)^2} + (y+1)^2$$

$$\le 3\sqrt{(x+2)^2+(y+1)^2}\cdot\sqrt{(x+2)^2+(y+1)^2}$$

$$+\ 3\sqrt{(x+2)^2+(y+1)^2} + 4\sqrt{(x+2)^2+(y+1)^2}$$

$$+\ (x+2)^2+(y+1)^2$$

$$= 3\left(\sqrt{(x+2)^2+(y+1)^2}\right)^2 + 7\sqrt{(x+2)^2+(y+1)^2}$$

$$+\left(\sqrt{(x+2)^2+(y+1)^2}\right)^2$$

$$= 4\left(\sqrt{(x+2)^2+(y+1)^2}\right)^2 + 7\sqrt{(x+2)^2+(y+1)^2}$$

assume $\delta \le 1$ $\le 4\sqrt{(x+2)^2+(y+1)^2} + 7\sqrt{(x+2)^2+(y+1)^2}$

so $A^2 \le A$ $= 11\sqrt{(x+2)^2+(y+1)^2} < \varepsilon$

iff $\sqrt{(x+2)^2+(y+2)^2} < \varepsilon/11$.

Now choose $\delta = \min\{1, \varepsilon/11\}$ and the result follows.

QED

g:) $\displaystyle\lim_{(x,y)\to(2,3)} \frac{8x-2y}{x+y} = 2$: Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that

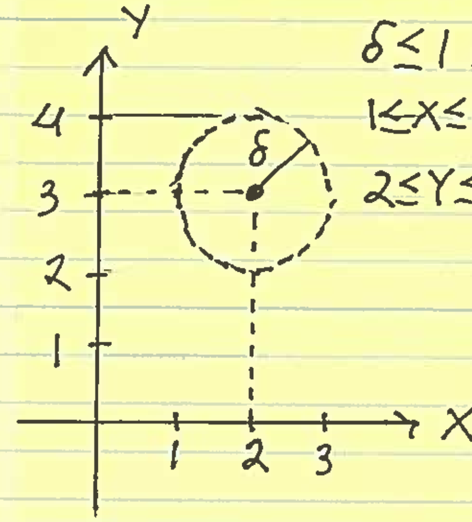
if $\sqrt{(x-2)^2+(y-3)^2} < \delta$, then $\left|\dfrac{8x-2y}{x+y} - 2\right| < \varepsilon$.

Then $\left|\dfrac{8x-2y}{x+y} - 2\right| = \left|\dfrac{8x-2y}{x+y} - 2\dfrac{(x+y)}{x+y}\right|$

$$= \left|\frac{8x-2y-2x-2y}{x+y}\right|$$

$$= \left| \frac{6X-4Y}{X+Y} \right| = \frac{|6X-4Y|}{|X+Y|}$$

(assume $\delta \le 1$
and play the
Big/Small Game.)



$\delta \le 1$ so
$1 \le X \le 3$,
$2 \le Y \le 4$

$$\le \frac{|6X-4Y|}{|(1)+(2)|}$$

$$= \frac{1}{3}|6X-4Y|$$

$$\le (1)\,|6X-4Y|$$

$$= |6(X-2)+\cancel{12}-4(Y-3)-\cancel{12}|$$

$$= |6(X-2)-4(Y-3)|$$

Δ-inequality

$$\le 6|X-2|+4|Y-3|$$

$$= 6\sqrt{(X-2)^2}+4\sqrt{(Y-3)^2}$$

$$\le 6\sqrt{(X-2)^2+(Y-3)^2}+4\sqrt{(X-2)^2+(Y-3)^2}$$

$$= 10\sqrt{(X-2)^2+(Y-3)^2} < \varepsilon$$

iff $\sqrt{(X-2)^2+(Y-3)^2} < \varepsilon/10$.

Choose $\delta = \min\{1, \varepsilon/10\}$ and the result follows.

QED