Math 21C  
Kouba  
Taylor Series and Taylor Polynomials

**QUESTION**: What connection do ordinary functions, \( y = f(x) \), have to power series centered at \( x = a \) of the form \( \sum_{n=0}^{\infty} a_n(x-a)^n \)?

**ANSWER**: Assume that \( y = f(x) \) is a given function and constant \( a \) is known. Determine a sequence of real numbers \( \{a_n\} \) so that

\[
(T) \quad f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots.
\]

If we substitute \( x = a \) in equation (T), we get

\[
f(a) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots = a_0,
\]
i.e.,

\[
a_0 = f(a).
\]

Now differentiate equation (T) term by term getting

\[
f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots.
\]

If we substitute \( x = a \) in this equation, we get

\[
f'(a) = a_1 + 2a_2(0) + 3a_3(0)^2 + 4a_4(0)^3 + \cdots = a_1,
\]
i.e.,

\[
a_1 = f'(a).
\]

Now differentiate again term by term getting

\[
f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 + 5 \cdot 4a_5(x-a)^3 + \cdots.
\]

If we substitute \( x = a \) in this equation, we get

\[
f''(a) = 2a_2 + 3 \cdot 2a_3(0) + 4 \cdot 3a_4(0)^2 + 5 \cdot 4a_5(0)^3 + \cdots = 2a_2,
\]
i.e.,

\[
a_2 = \frac{f''(a)}{2!}.
\]

Now differentiate again term by term getting

\[
f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-a) + 5 \cdot 4 \cdot 3a_5(x-a)^2 + 6 \cdot 5 \cdot 4a_6(x-a)^3 + \cdots.
\]

If we substitute \( x = a \) in this equation, we get

\[
f'''(a) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(0) + 5 \cdot 4 \cdot 3a_5(0)^2 + 6 \cdot 5 \cdot 4a_6(0)^3 + \cdots = 3 \cdot 2a_3,
\]
i.e.,

\[
a_3 = \frac{f'''(a)}{3!}.
\]
Continuing this term by term differentiation and substitution process results in the fact that

(S) \[ a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n = 0, 1, 2, 3, 4, \ldots. \]

**DEFINITION**: Equations (T) and (S) together are called the *Taylor Series* for function \( y = f(x) \) centered at \( x = a \). For the special case of \( a = 0 \), we call the series a *Maclaurin Series*.

**DEFINITION**: The *Taylor Polynomial of degree n* centered at \( x = a \) for function \( y = f(x) \) is given by

(P) \[ P_n(x; a) = \sum_{k=0}^{n} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n \]

and

(S) \[ a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, 2, 3, \ldots, n. \]

**REMARK**: The *Taylor Polynomial of degree n* centered at \( x = a \) for function \( y = f(x) \) is the *Taylor Series* centered at \( x = a \) terminated at the \( n \)-th power of \( (x-a) \). It is NOT defined to be the first \( n \) or \( n+1 \) terms of the Taylor Series.

**QUESTION**: For what \( x \)-values is an ordinary function \( y = f(x) \) equal to its Taylor series centered at \( x = a \), i.e., for what \( x \)-values is \( y = f(x) \) equal to

\[ f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots \]

**ANSWER**: Let \( P_n(x; a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \) be the *Taylor Polynomial of degree n* centered at \( x = a \) and let

\[ R_n(x; a) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots, \]

which is simply the remaining infinite tail of the Taylor Series centered at \( x = a \). It can be shown that

\[ R_n(x; a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}, \]

where \( c_n \) is between numbers \( a \) and \( x \). This is called the Lagrange form of the Taylor remainder. Those \( x \)-values for which \( y = f(x) \) is equal to its Taylor series are precisely those \( x \)-values for which

\[ \lim_{n \to \infty} R_n(x; a) = 0. \]