Math 21C (Fall 2017)
Kouba
Exam 2

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE THIS EXAM FOR YOU. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 9:50 a.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. Make sure that you have 7 pages including the cover page.

1.) (9 pts.) Determine the interval of convergence (including endpoints) for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x+2)^n; \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|x+2|^n}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{3} \cdot |x+2| = 1 \cdot \frac{1}{3} \cdot |x+2| < 1 \rightarrow |x+2| < 3$$

$$\rightarrow -3 < x+2 < 3 \rightarrow \underline{-5 < x < 1};$$

If $\underline{x=-5}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$

$\underline{\text{diverges}}$ by p-series $(p = 1 \leq 1)$;

If $\underline{x=1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ $\underline{\text{converges}}$

by A.S.T. since $\frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n} = 0$,

interval of convergence: $\boxed{-5 < x \leq 1}$

2.) (9 pts.) Use $a_n = \dfrac{f^{(n)}(a)}{n!}$ to find the first three (3) nonzero terms of the Taylor Series centered at $x = -2$ for $f(x) = (x+3)^{10}$.

$$\xrightarrow{D} f'(x) = 10(x+3)^9 \xrightarrow{D} f''(x) = 90(x+3)^8,$$

$$a_0 = f(-2) = (1)^{10} = 1, \quad a_1 = f'(-2) = 10(1)^9 = 10,$$

$$a_2 = \frac{f''(-2)}{2!} = \frac{90}{2}(1)^8 = 45, \quad \text{so}$$

$$(x+3)^{10} = 1 + 10(x+2) + 45(x+2)^2 + \cdots$$

3.) (9 pts.) Use shortcuts to find the first three (3) nonzero terms of the Maclaurin Series for $f(x) = \dfrac{e^{x^2}}{1+x}$.

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \quad \text{and}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots, \quad \text{then}$$

$$\frac{e^{x^2}}{1+x} = e^{x^2} \cdot \frac{1}{1+x} = \left(1 + x^2 + \frac{x^4}{2} + \cdots\right)\left(1 - x + x^2 - x^3 + \cdots\right)$$

$$= 1 - x + x^2 - x^3 + \cdots$$
$$+ x^2 - x^3 + \cdots$$

$$= 1 - x + 2x^2 - \cdots$$

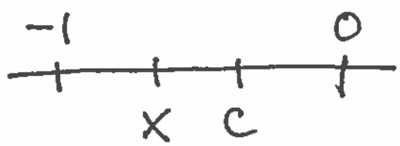
4.) (9 pts.) Consider the function $f(x) = \dfrac{x}{x-3}$ on the closed interval $[-1, 0]$. Estimate the value of $\left|R_3(x;1)\right|$, the absolute value of the Taylor Remainder (error).

$$\xrightarrow{D} f'(x) = \frac{x-3-x}{(x-3)^2} = -3(x-3)^{-2} \xrightarrow{D} f''(x) = 6(x-3)^{-3}$$

$$\xrightarrow{D} f'''(x) = -18(x-3)^{-4} \xrightarrow{D} f^{(4)}(x) = 72(x-3)^{-5};$$

$n = 3$, $a = 1$, $[-1, 0]$, so

$$|R_3(x;1)| = \left|\frac{f^{(4)}(c)}{4!}(x-1)^4\right| = \frac{72}{4!} \cdot \frac{|x-1|^4}{|c-3|^5}$$



$$\leq 3 \cdot \frac{|(-1)-1|^4}{|(0)-3|^5}$$

$$= \frac{16}{81} \approx 0.1975$$

5.) (9 pts.) Let $\vec{u} = \overrightarrow{(2,-1,3)}$ and $\vec{v} = \overrightarrow{(1,0,-2)}$. Find $proj_{\vec{v}}\, \vec{u}$.

$$proj_{\vec{v}}\, \vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{2+0-6}{1+0+4}\right)\vec{v}$$

$$= -\frac{4}{5}\,\overrightarrow{(1,0,-2)}$$

$$= \overrightarrow{\left(-\frac{4}{5}, 0, \frac{8}{5}\right)}$$

6.) (10 pts.) Use a Taylor Polynomial to estimate the value of $\int_0^{1/5} \frac{1-\cos x}{x^2}\, dx$ with absolute error at most 0.0001.

$$\int_0^{1/5} \frac{1-\cos x}{x^2}\,dx = \int_0^{1/5} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots}{x^2}\,dx$$

$$= \int_0^{1/5} \left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right) dx$$

$$= \left(\frac{1}{2!}x - \frac{x^3}{4!\,3} + \frac{x^5}{6!\,5} - \frac{x^7}{8!\,7} + \cdots\right)\Big|_0^{1/5}$$

$$= \frac{1}{2!}\left(\frac{1}{5}\right) - \frac{\left(\frac{1}{5}\right)^3}{4!\,3} + \frac{\left(\frac{1}{5}\right)^5}{6!\,5} - \cdots \quad \text{(Use A.S.T. error)}$$

< 0.0001

$$\approx \frac{1}{2!}\left(\frac{1}{5}\right) - \frac{\left(\frac{1}{5}\right)^3}{4!\,3}$$

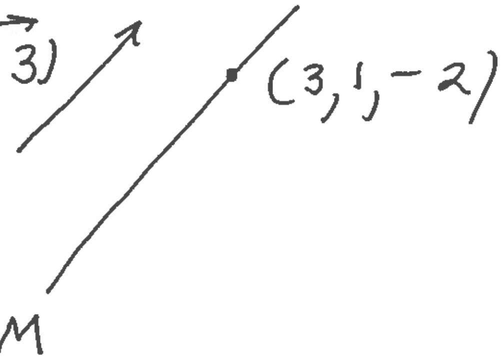
$$\approx 0.09988$$

7.) (9 pts.) Find the parametric equations for the line passing through the point $(3, 1, -2)$ and which is parallel to the line given by

$$L: \begin{cases} x = 1 + 2t \\ y = 4 - t \\ z = 3t - 5 \end{cases}$$

$\vec{A} = \overrightarrow{(2, -1, 3)}$

Direction Vector



so line is

$$M: \begin{cases} X = 3 + 2t \\ Y = 1 - t \\ Z = -2 + 3t \end{cases}$$

8.) (9 pts.) Determine parametric equations for the line L representing the intersection of the planes $x - y + 2z = 0$ and $3x + y - z = 1$.

$$\begin{cases} X - Y + 2Z = 0 \\ 3X + Y - Z = 1 \end{cases} \xrightarrow{\text{(ADD)}} 4X + Z = 1 \longrightarrow Z = 1 - 4X$$

so let $X = t$ any #, $Z = 1 - 4t$, and

$Y = X + 2Z = t + 2(1 - 4t) = 2 - 7t$ so

$$\text{line } L: \begin{cases} X = t \\ Y = 2 - 7t \\ Z = 1 - 4t \end{cases}$$

9.) (9 pts.) The points $(0,0,0)$, $(1,0,-1)$, and $(0,1,2)$ form a triangle in three-dimensional space. Determine the area of this triangle.

Form vectors $\vec{A} = \overrightarrow{(1-0, 0-0, -1-0)} = \overrightarrow{(1,0,-1)}$,

$\vec{B} = \overrightarrow{(0-0, 1-0, 2-0)} = \overrightarrow{(0,1,2)}$;

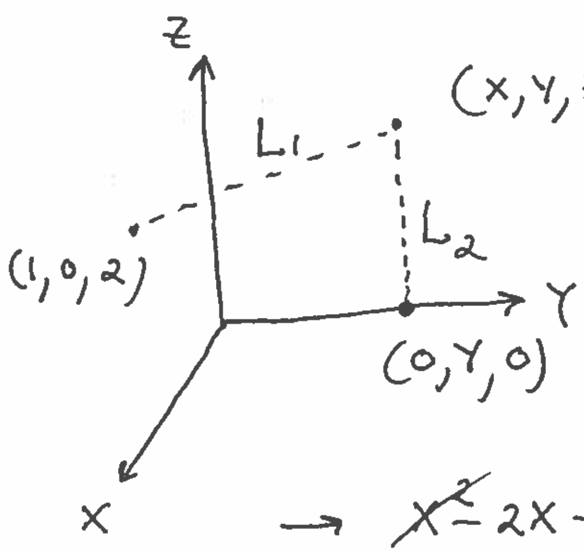
$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = (0+1)\vec{i} - (2-0)\vec{j} + (1-0)\vec{k}$$

$$= \vec{i} - 2\vec{j} + \vec{k}$$, then

area of Δ is

$$\frac{1}{2}|\vec{A} \times \vec{B}| = \frac{1}{2}\sqrt{1+4+1} = \frac{\sqrt{6}}{2}$$

10.) (9 pts.) Find an equation for the set of all points (x, y, z) equidistant from point $(1, 0, 2)$ and the y-axis. Simplify your answer as much as possible.



$L_1 = L_2 \rightarrow$

$$\sqrt{(X-1)^2 + (Y-0)^2 + (Z-2)^2}$$

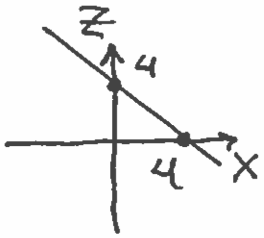
$$= \sqrt{(X-0)^2 + (Y-Y)^2 + (Z-0)^2}$$

$$\rightarrow \cancel{X^2} - 2X + 1 + Y^2 + \cancel{Z^2} - 4Z + 4$$

$$= \cancel{X^2} + 0 + \cancel{Z^2} \rightarrow$$

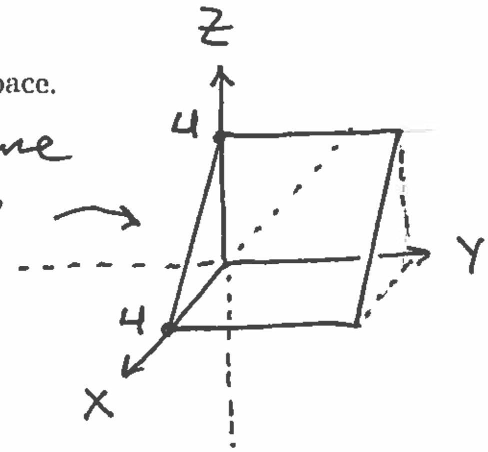
$$Y^2 + 5 = 2X + 4Z$$

11.) a.) (5 pts.) Sketch the graph of $x + z = 4$ in 3D-Space.



Tilted Plane

$x + z = 4 \rightarrow$



b.) (4 pts.) Find the distance between the points $(1, 4, 3)$ and $(-1, 1, -3)$.

$$\text{Dist.} = \sqrt{(1-(-1))^2 + (4-1)^2 + (3-(-3))^2}$$
$$= \sqrt{4 + 9 + 36}$$
$$= \sqrt{49}$$
$$= 7$$

The following EXTRA CREDIT PROBLEM is worth 8 points. This problem is OPTIONAL.

1.) Find the exact value of $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 3^n}{n!} = -3 + \frac{2^2 \cdot 3^2}{2!} - \frac{3^2 \cdot 3^3}{3!} + \frac{4^2 \cdot 3^4}{4!} - \cdots$

$$-e^{-x} = -\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = -1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots \xrightarrow{D}$$

$$e^{-x} = 1 - \frac{2x}{2!} + \frac{3x^2}{3!} - \frac{4x^3}{4!} + \cdots \rightarrow (\text{mult. } x)$$

$$x e^{-x} = x - \frac{2x^2}{2!} + \frac{3x^3}{3!} - \frac{4x^4}{4!} + \cdots \xrightarrow{D}$$

$$e^{-x} - x e^{-x} = 1 - \frac{2^2 x}{2!} + \frac{3^2 x^2}{3!} - \frac{4^2 x^3}{4!} + \cdots \rightarrow (\text{mult. } x)$$

$$x^2 e^{-x} - x e^{-x} = -x + \frac{2^2 x^2}{2!} - \frac{3^2 x^3}{3!} + \frac{4^2 x^4}{4!} - \cdots \quad \left(\text{let } x = 3\right)$$

$$9e^{-3} - 3e^{-3} = \boxed{6e^{-3}} = -3 + \frac{2^2 3^2}{2!} - \frac{3^2 3^3}{3!} + \frac{4^2 3^4}{4!} - \cdots$$