Math 21C (Spring 2015)
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ___________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. HAVING ANOTHER STUDENT TAKE YOUR EXAM FOR YOU IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 8:50 a.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. Make sure that you have 9 pages including the cover page.

1.) (8 pts.) Let $z = y\tan x + 3e^{2y}$. Compute the partial derivatives z_x, z_y, z_{xy}, and z_{yy} . You need NOT SIMPLIFY your answers.

$$z_x = y\sec^2 x + 0$$

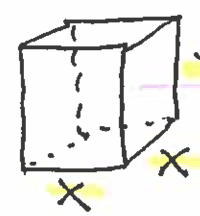
$$z_y = \tan x + 3e^{2y} \cdot 2$$

$$z_{yy} = 12e^{2y}$$

$$z_{xy} = \sec^2 x$$

2.) (8 pts.) A closed rectangular box has a square base, x feet by x feet, and height y feet. In addition assume that x is increasing at the rate of 2 ft./min. and the box's surface area S is increasing at the rate of 32 ft.^{2}/min. Determine the rate at which y is changing when $x = 4$ feet and $y = 5$ feet.

$\frac{dx}{dt} = 2$ ft./min. , surface area



$S = 2x^2 + 4xy$, $\frac{dS}{dt} = 32 \frac{\text{ft.}^2}{\text{min.}}$ $\xrightarrow{D}$

$$\frac{dS}{dt} = 4x \cdot \frac{dx}{dt} + 4x \cdot \frac{dy}{dt} + 4\frac{dx}{dt} \cdot y \rightarrow (x=4, y=5)$$

$$32 = 4(4)(2) + 4(4) \cdot \frac{dy}{dt} + 4(2)(5) \rightarrow$$

$$32 = 32 + 16\frac{dy}{dt} + 40 \longrightarrow 16\frac{dy}{dt} = -40 \longrightarrow$$

$$\frac{dy}{dt} = -\frac{40}{16} = \boxed{\frac{-5}{2} \text{ ft./min.}}$$

3.) (8 pts.) Evaluate the following limit : $\lim_{(x,y)\to(2,-3)} \dfrac{xy - 3x - 2y + 6}{x^2 - 4}$

"$\frac{0}{0}$"

$$\lim_{(x,y)\to(2,-3)} \frac{\cancel{(x-2)}(y-3)}{\cancel{(x-2)}(x+2)}$$

$$= \frac{-6}{4}$$

$$= -\frac{3}{2}$$

4.) (8 pts.) Verify that the following limit does not exist : $\lim_{(x,y)\to(0,0)} \dfrac{y \cdot \sin x}{\sin^2 x + y^2}$

<u>Path $y=0$</u>: $\lim_{(x,y)\to(0,0)} \dfrac{0}{\sin^2 x + 0} = \lim_{(x,y)\to(0,0)} 0 = 0$

<u>Path $y = \sin x$</u>:

$$\lim_{(x,y)\to(0,0)} \frac{\sin^2 x}{\sin^2 x + \sin^2 x} = \lim_{(x,y)\to(0,0)} \frac{\cancel{\sin^2 x}}{2\cancel{\sin^2 x}}$$

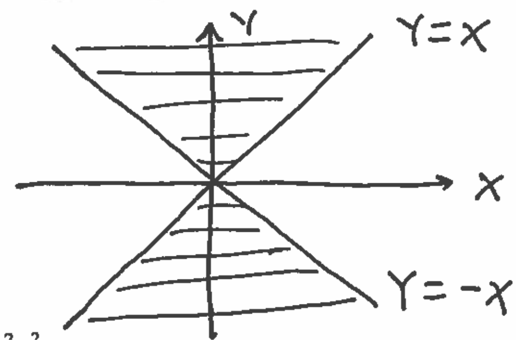
$$= \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

so $\lim_{(x,y)\to(0,0)} \dfrac{y \sin x}{\sin^2 x + y^2}$ DNE

5.) a.) (5 pts.) Let $f(x,y) = \sqrt{y^2 - x^2}$. Determine and sketch the domain of f in 2D-space.

$Y^2 - X^2 = (Y-X)(Y+X) \geq 0$,

so Domain: all pts. (X,Y) on lines $Y = X$, $Y = -X$, and shaded area



b.) (5 pts.) Determine the range of $f(x,y) = 3 + e^{-x^2y^2}$.

$$\max_{-\infty < X,Y < \infty} e^{-X^2Y^2} = e^{-(0)(0)} = 1,$$

$$\lim_{X,Y \to \pm\infty} e^{-X^2Y^2} = e^{-\infty} = 0, \text{ so}$$

Range: $3 < Z \leq 4$

6.) (8 pts.) Consider the function $f(x,y) = x^2 - y^3 + 2xy$ and the point $(3,-1)$. What is the maximum value of $D_{\vec{u}}f(3,-1)$ and in what direction ?

$\vec{\nabla} f = \overrightarrow{(2X+2Y, -3Y^2+2X)}$ so

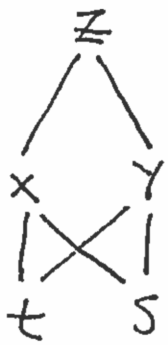
$\vec{\nabla} f(3,-1) = \overrightarrow{(4,3)}$, then MAX

$$D_{\vec{u}} f(3,-1) = |\vec{\nabla} f(3,-1)|$$

$$= |\overrightarrow{(4,3)}| = \sqrt{4^2+3^2} = 5 \; ;$$

in direction of the gradient vector $\overrightarrow{(4,3)}$.

7.) (9 pts.) Use the Chain Rule to find and simplify (as much as possible) the derivatives $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial^2 z}{\partial t^2}$, where $z = f(x, y)$, $x = t - e^s$, and $y = s \ln t$.



$$\frac{\partial z}{\partial t} = f_x \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t}$$

$$= f_x \cdot (1) + f_y \cdot \left(\frac{s}{t}\right) \text{, then}$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial t}\left(f_x + f_y \cdot \frac{s}{t}\right)$$

$$= f_{xx} \frac{\partial x}{\partial t} + f_{xy} \cdot \frac{\partial y}{\partial t}$$

$$+ \left(f_y \cdot \frac{-s}{t^2} + \left(f_{yx} \frac{\partial x}{\partial t} + f_{yy} \cdot \frac{\partial y}{\partial t}\right) \cdot \frac{s}{t}\right)$$

$$= f_{xx} \cdot (1) + f_{xy} \cdot \frac{s}{t} - f_y \cdot \frac{s}{t^2}$$

$$+ f_{xy} (1) \cdot \frac{s}{t} + f_{yy} \cdot \left(\frac{s}{t}\right)\left(\frac{s}{t}\right)$$

$$= f_{xx} + \frac{2s}{t} \cdot f_{xy} - \frac{s}{t^2} \cdot f_y + \frac{s^2}{t^2} \cdot f_{yy}$$

8.) Consider the function given by $z = 6 - 3x - 2y$ and its graph in 3D-space.

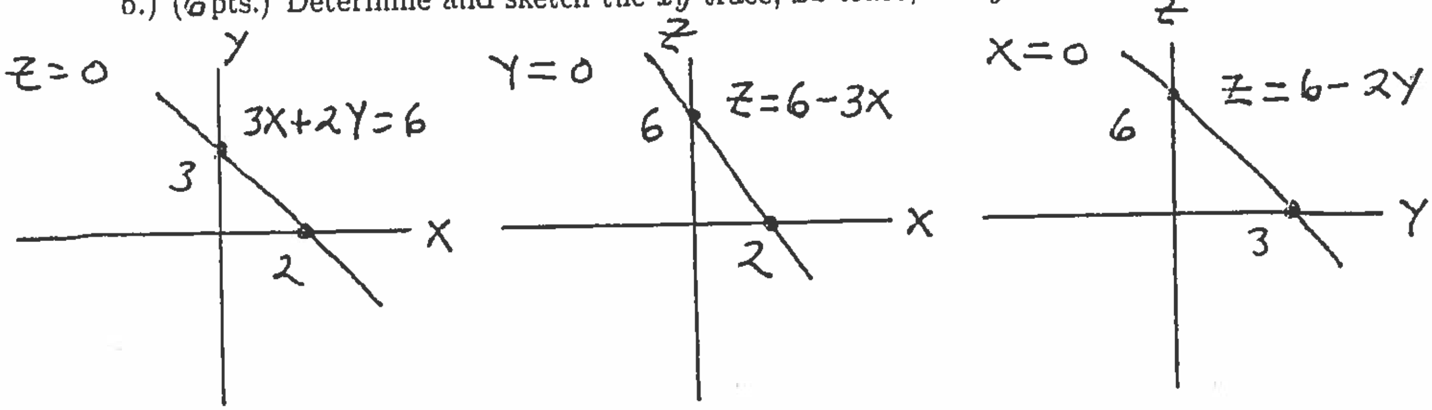
a.) (3 pts.) Determine all possible intercepts for this equation.

$x=0, y=0: \quad z = 6$

$x=0, z=0: \quad 0 = 6 - 2y \rightarrow y = 3$

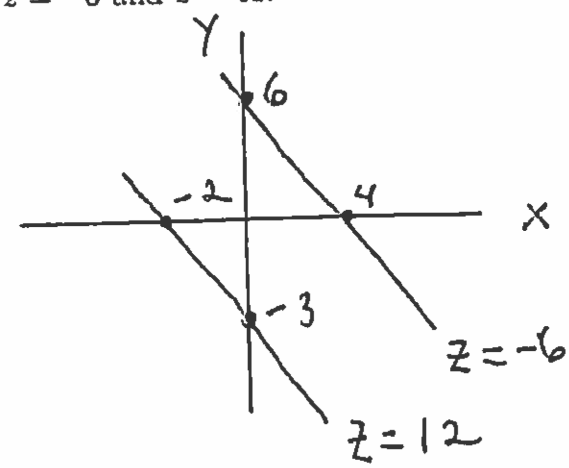
$y=0, z=0: \quad 0 = 6 - 3x \rightarrow x = 2$

b.) (6 pts.) Determine and sketch the xy-trace, xz-trace, and yz-trace.

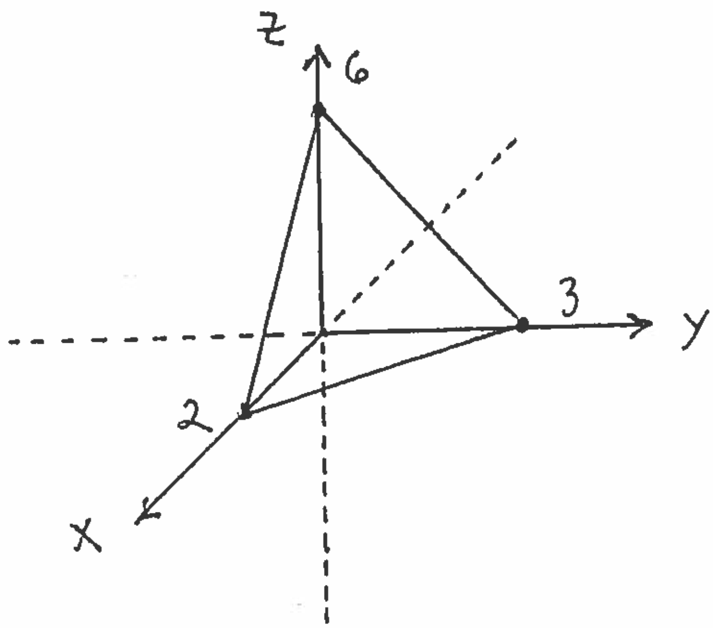


c.) (4 pts.) Sketch level curves on the same axes for $z = -6$ and $z = 12$.

z	level curve
-6	$-6 = 6 - 3x - 2y \rightarrow$ $3x + 2y = 12$
12	$12 = 6 - 3x - 2y \rightarrow$ $3x + 2y = -6$



d.) (3pts.) Sketch the surface in 3D-space.



9.) (9 pts.) Write an ϵ, δ-proof for $\displaystyle\lim_{(x,y)\to(-1,1)} (x^2 + y) = 2.$

Let $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < \sqrt{(x+1)^2 + (y-1)^2} < \delta$, then $|x^2 + y - 2| < \varepsilon$.

Then

$$|x^2+y-2| = |(x+1)^2 - 2x - \cancel{1} + (y-1) + \cancel{1} - 2|$$

$$= |(x+1)^2 - 2(x+1) + (y-1)|$$

$$\le (x+1)^2 + 2|x+1| + |y-1|$$

$$= (x+1)^2 + 2\sqrt{(x+1)^2} + \sqrt{(y-1)^2}$$

$$\le (x+1)^2 + (y-1)^2 + 2\sqrt{(x+1)^2+(y-1)^2} + \sqrt{(x+1)^2+(y-1)^2}$$

$$= \left(\sqrt{(x+1)^2 + (y-1)^2}\right)^2 + 3\sqrt{(x+1)^2 + (y-1)^2}$$

assume $\delta \le 1$

$$\le \sqrt{(x+1)^2 + (y-1)^2} + 3\sqrt{(x+1)^2 + (y-1)^2}$$

$$= 4\sqrt{(x+1)^2 + (y-1)^2} < \varepsilon$$

iff $\sqrt{(x+1)^2 + (y-1)^2} < \varepsilon/4$. Now choose $\delta = \min\{1, \varepsilon/4\}$ and the result follows.

Q.E.D.

10.) (8 pts.) Consider the function $f(x, y) = x^2y$ and the point $P = (-1, 1)$. Find all possible vectors $\vec{u}$ so that the directional derivative $D_{\vec{u}}f(-1, 1) = 2$.

Let $\vec{u} = \overrightarrow{(a,b)}$ where $\boxed{a^2 + b^2 = 1}$;

$\vec{\nabla} f = \overrightarrow{(2xy, x^2)}$ then $\vec{\nabla} f(-1,1) = \overrightarrow{(-2,1)}$, and

$D_{\vec{u}} f(-1,1) = \vec{\nabla} f(-1,1) \cdot \vec{u} = \overrightarrow{(-2,1)} \cdot \overrightarrow{(a,b)}$

$= -2a + b = 2 \rightarrow \boxed{b = 2a + 2}$; then

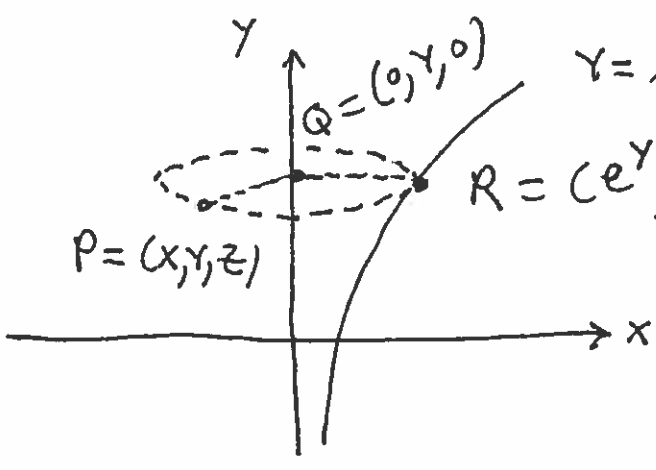
$a^2 + (2a+2)^2 = 1 \rightarrow a^2 + 4a^2 + 8a + 4 = 1 \rightarrow$

$5a^2 + 8a + 3 = 0 \rightarrow (5a+3)(a+1) = 0$

$\rightarrow a = -3/5$, $b = -\frac{6}{5} + \frac{10}{5} = \frac{4}{5} \rightarrow \boxed{\vec{u} = (-3/5, 4/5)}$ or

$\rightarrow a = -1$, $b = 0 \rightarrow \boxed{\vec{u} = \overrightarrow{(-1,0)}}$

11.) (8 pts.) Consider the graph of $y = \ln x$ in the xy-plane. Find an equation for the surface created by revolving this graph about the y-axis.



$Y = \ln X$ or $X = e^Y$

$\sqrt{(X-0)^2 + (Y-Y)^2 + (z-0)^2}$

$= \sqrt{(e^Y - 0)^2 + (Y-Y)^2 + (0-0)^2}$

$\rightarrow X^2 + z^2 = e^{2Y}$

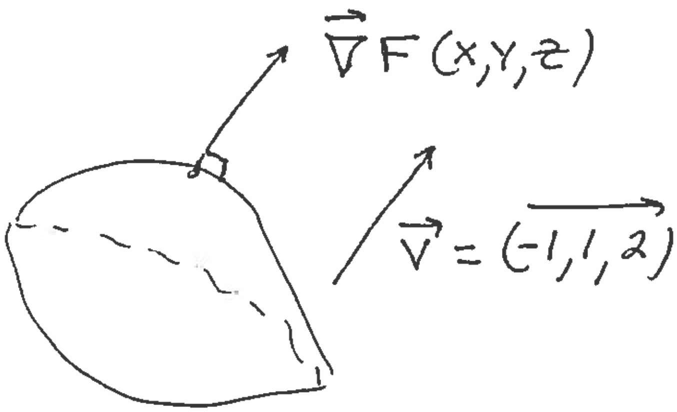
The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Find all points (x, y, z) on the hyperboloid $z^2 = x^2 - xy + y^2 + 6$ which have normal (perpendicular) vectors parallel to the vector $\vec{v} = \overrightarrow{(-1, 1, 2)}$.

$$0 = \underbrace{x^2 - xy + y^2 + 6 - z^2}_{F(x,y,z)}$$

$\vec{\nabla} F \parallel \vec{v}$ so

$\vec{\nabla} F \times \vec{v} = \vec{0}$;



$$\vec{\nabla} F(x,y,z) = \overrightarrow{(2x-y,\ 2y-x,\ -2z)} \text{, then}$$

$$\vec{\nabla} F \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2x-y & 2y-x & -2z \\ -1 & 1 & 2 \end{vmatrix}$$

$$= \left(2(2y-x) - (-2z)\right)\vec{i} - \left(2(2x-y) - 2z\right)\vec{j} + \left(2x-y + (2y-x)\right)\vec{k}$$

$$= (4y - 2x + 2z)\vec{i} - (4x - 2y - 2z)\vec{j} + (x+y)\vec{k} = \vec{0}$$

$$\rightarrow x + y = 0 \rightarrow \boxed{y = -x} \rightarrow 4(-x) - 2x + 2z = 0$$

$$\rightarrow 2z = 6x \rightarrow \boxed{z = 3x} \rightarrow (\text{SUB})$$

$$(3x)^2 = x^2 - x(-x) + (-x)^2 + 6 \rightarrow 6x^2 = 6 \rightarrow x = \pm 1$$

$$\rightarrow \boxed{(1,-1,3)} \text{ and } \boxed{(-1,1,-3)} .$$