Section 13.1
Thomas Calculus
11th Ed.

What Does The Acceleration Vector Represent?

We know that a vector function $\vec{r}(t)$ creates a path C in space (2D or 3D). It's velocity vector $\vec{r}\,'(t) = \vec{v}(t)$ satisfies:

1.) $\vec{v}(t)$ points in the direction of motion along path C.
2.) $\vec{v}(t)$ is tangent to path C.
3.) $|\vec{v}(t)|$ is the speed of motion along path C.

What does the acceleration vector $\vec{r}\,''(t) = \vec{v}\,'(t) = \vec{a}(t)$

represent? Let's look at an example.

Example: Let vector function

$$\vec{r}(t) = (t^2)\vec{i} + \frac{1}{2}(t^4 - t^2)\vec{j} \quad , t \geq 0.$$

Plot the path C determined by $\vec{r}(t)$. Find and plot $\vec{r}(1)$. Find and plot $\vec{v}(1)$ and $\vec{a}(1)$ at the point determined by $\vec{r}(1)$:

$$\vec{r}(t) = (t^2)\vec{i} + \frac{1}{2}(t^4 - t^2)\vec{j} \xrightarrow{D}$$

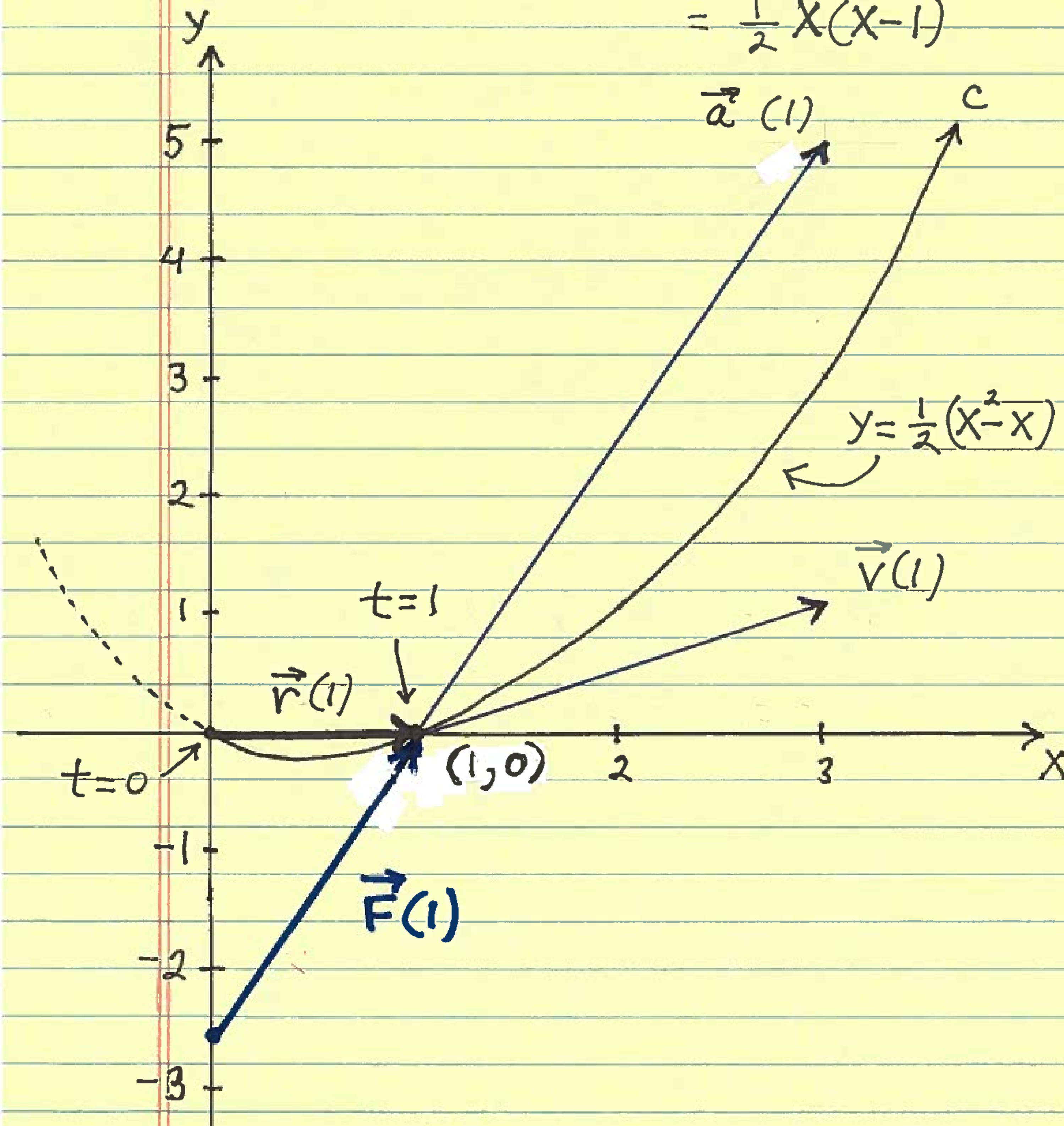
$$\vec{v}(t) = \vec{r}\,'(t) = (2t)\vec{i} + (2t^3 - t)\vec{j} \xrightarrow{D}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = (2)\vec{i} + (6t^2 - 1)\vec{j}, \text{ then}$$

$\vec{r}(1) = (1)\vec{i} + (0)\vec{j}$, $\vec{v}(1) = (2)\vec{i} + (1)\vec{j}$, and $\vec{a}(1) = (2)\vec{i} + (5)\vec{j}$.

Find path C: $\begin{cases} x = t^2 \\ y = \frac{1}{2}(t^4 - t^2) \end{cases} \rightarrow$

$$y = \frac{1}{2}\left((t^2)^2 - (t^2)\right) = \frac{1}{2}(x^2 - x)$$
$$= \frac{1}{2}x(x-1)$$



Recall that the definition of FORCE is an action that changes the motion of an object, e.g., gravity, wind, magnetism, love, water pressure, etc.

Assume an object of mass m is moving through space in a straight line at a constant speed. To change the speed or direction of the object requires the action of an outside vector force $\vec{F}$. According to Newton's 2nd Law of Motion, this force $\vec{F}$, the mass m of the object, and the object's acceleration vector $\vec{a}$ must satisfy

$$\vec{F}(t) = m\,\vec{a}(t) .$$

So now we can conclude

that the acceleration vector $\vec{a}(t)$ points in the direction of the force required to change the direction and speed of the object of mass m and keep it moving along path C. And the magnitude of this force vector is

$$\left|\vec{F}(t)\right| = m\left|\vec{a}(t)\right| .$$

Let's assume that an object of mass $m = \frac{1}{2}$ kg is traveling along our path C. The force vector at point $(1,0)$ required to change the speed and direction of this object is

$$\vec{F}(1) = \left(\tfrac{1}{2}\right)\left(2\vec{i} + 5\vec{j}\right)$$
$$= \vec{i} + \tfrac{5}{2}\vec{j}, \quad \text{and}$$

its magnitude (Let's assume distance is measured in meters) is

$$|\vec{F}(1)| = \sqrt{(1)^2 + \left(\frac{5}{2}\right)^2}$$

$$= \sqrt{\frac{4}{4} + \frac{25}{4}}$$

$$= \frac{\sqrt{29}}{2} \text{ Newtons.}$$

How is the acceleration vector related to the acceleration of motion along path C?

Recall that the position, velocity, and acceleration vectors are given by

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k} \xrightarrow{D}$$

$$\vec{v}(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k} \xrightarrow{D}$$

$$\vec{a}(t) = x''(t)\,\vec{i} + y''(t)\,\vec{j} + z''(t)\,\vec{k} \ ;$$

and speed of motion is

$$\frac{ds}{dt} = \left|\vec{v}(t)\right| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2},$$

so acceleration of motion is

$$a(t) = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

$$= \frac{1}{2}\left((x'(t))^2 + (y'(t))^2 + (z'(t))^2\right)^{-1/2} \cdot$$

$$\left(2x'(t)\cdot x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t)\right)$$

DOT PRODUCT

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{|\vec{v}(t)|} \cdot \cancel{2}\left(\vec{v}(t) \bullet \vec{a}(t)\right) \text{, i.e.,}$$

$$\boxed{a(t) = \frac{\vec{v}(t) \bullet \vec{a}(t)}{|\vec{v}(t)|}}$$

is the acceleration of motion along path C.