Section 13.3
Thomas Calculus
11th Ed.

Arc Length in Space

Consider path C in 3D-space traced out by the position vector

\[ \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \]

velocity vector

\[ \mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}, \]

acceleration vector

\[ \mathbf{a}(t) = \mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}, \]

and speed of motion

\[ \frac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}. \]

Definition: The arc length of path C from \( t=a \) to \( t=b \) is
\[ S(t) = \int_a^t \sqrt{(f'(\tau))^2 + (g'(\tau))^2 + (h'(\tau))^2} \, d\tau \]

\[ = \int_a^t |\vec{v}(\tau)| \, d\tau \]

**Example:** Consider path \( C \) in 2D-space determined by the position vector

\[ \vec{r}(t) = (t^2)\vec{i} + (\frac{2}{3}t^3)\vec{j} \]
First find the arc length of $C$ from $t=1$ to $t$. Then find the arc length from $t=1$ to $t=4$.

\[
s(t) = \int_1^t \sqrt{(2e^t)^2 + (2e^t)^2} \, dt
\]
\[
= \int_1^t \sqrt{4e^{2t} + 4e^{2t}} \, dt
\]
\[
= \int_1^t \sqrt{4e^{2t}(1 + e^{2t})} \, dt
\]
\[
= \int_1^t 2e^{t} \sqrt{1 + e^{2t}} \, dt
\]
\[
= \frac{2}{3} \left(1 + e^{2t}\right)^{3/2} \bigg|_1^t
\]
\[
= \frac{2}{3} \left(1 + t^2\right)^{3/2} - \frac{2}{3} \left(2\right)^{3/2}, \text{ i.e.,}
\]

Arc length is

\[
s(t) = \frac{2}{3} \left(1 + t^2\right)^{3/2} - \frac{2}{3} \left(2\right)^{3/2}.
\]
Arc Length from $t=1$ to $t=4$ is

$$s(4) = \frac{2}{3} (17)^{3/2} - \frac{2}{3} (2)^{3/2} \approx 44.84$$

**Example:** Consider path $C$ in 3D-space determined by the position vector

$$\vec{r}(t) = (2\sin 2t) \hat{i} + (2\cos 2t) \hat{j} + (3t) \hat{k}.$$  

First find the arc length of $C$ from $t=0$ to $t$. Then find the arc length from $t=0$ to $t=2$.

$$s(t) = \int_0^t \sqrt{(4\cos 2t)^2 + (-4\sin 2t)^2 + (3)^2} \, dt$$

$$= \int_0^t \sqrt{16\cos^2 2t + 16\sin^2 2t + 9} \, dt$$

$$= \int_0^t \sqrt{16 \cos^2 2t + \sin^2 2t + 9} \, dt$$

$$= \int_0^t \sqrt{16(1) + 9} \, dt$$
\[ = \int_0^t \sqrt{25} \, dt = \int_0^t 5 \, dt \]
\[ = 5t \bigg|_0^t = 5t \], i.e., arc length is \[ s(t) = 5t \].

Arc length from \( t=0 \) to \( t=2 \) is \[ s(2) = 5(2) = 10 \]
What is the origin of the definition of arc length?

Consider finding the distance $L$ between two points $A$ and $B$ in 3D-space. Form a rectangular box of dimensions $dx$, $dy$, and $dz$ and with $L$ as its main diagonal:

\[ \begin{align*}
&\text{Diagram of a rectangular box with points } A, M, M', N, B \\
&\text{and vectors } dx, dy, dz. \\
&\end{align*} \]
By the Pythagorean Theorem

\[(dx)^2 + (dy)^2 = M^2 \text{ and } M^2 + (dz)^2 = L^2 \rightarrow\]

\[L^2 = (dx)^2 + (dy)^2 + (dz)^2 \rightarrow\]

\[L = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}.\]

Let's apply this idea to arc length on path C. Consider a small piece of arc length \(ds\) determined by \(dx, dy,\) and \(dz\), small changes in \(x, y,\) and \(z\).

The arc length of C from \(t = a\)
to \( t \) is

\[
s(t) = \int ds
\]

\[
= \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2} d\tau
\]

\[
= \int_a^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2} d\tau
\]

i.e., arc length is

\[
s(t) = \int_a^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2} d\tau.
\]