Section 13.3
Thomas Calculus
11th Ed.

Arc Length in Space

Consider path C in 3D-Space traced out by the position vector

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k},$$

velocity vector

$$\vec{v}(t) = \vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k},$$

acceleration vector

$$\vec{a}(t) = \vec{r}\,''(t) = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k},$$

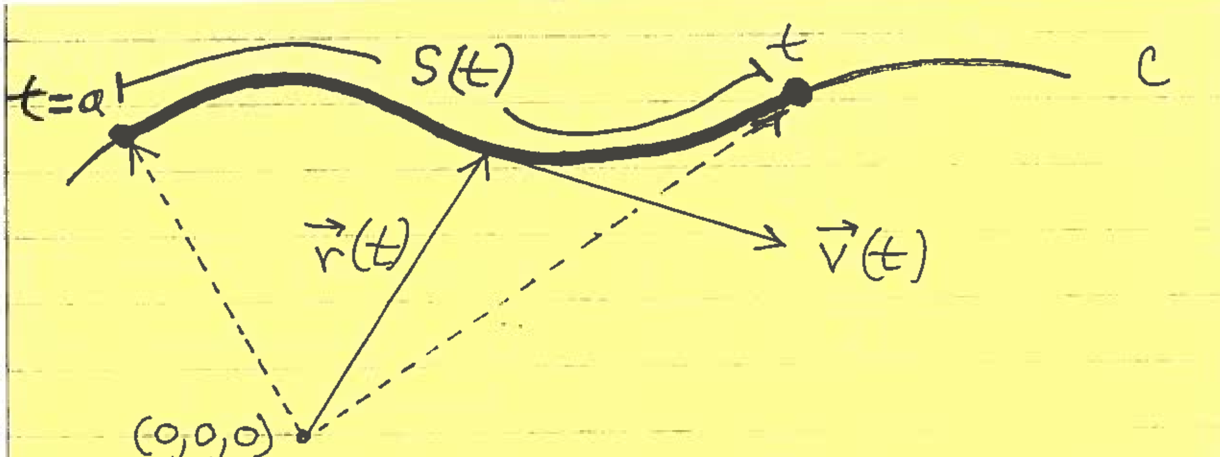
and speed of motion

$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}.$$

<u>Definition</u>: The <u>arc</u> <u>length</u> of path C from $t=a$ to t is

$$s(t) = \int_a^t \sqrt{(f'(\tau))^2 + (g'(\tau))^2 + (h'(\tau))^2}\, d\tau$$

$$= \int_a^t |\vec{v}(\tau)|\, d\tau$$



$$s(t) = \int_a^t \sqrt{(f'(\tau))^2 + (g'(\tau))^2 + (h'(\tau))^2}\, d\tau$$

$$= \int_a^t |\vec{v}(\tau)|\, d\tau .$$

<u>Example</u>: Consider path C in 2D-Space determined by the position vector

$$\vec{r}(t) = (t^2)\vec{i} + \left(\frac{2}{3}t^3\right)\vec{j} .$$

First find the arc length of C from $t=1$ to t. Then find the arc length from $t=1$ to $t=4$.

$$s(t) = \int_1^t \sqrt{(2\tau)^2 + (2\tau^2)^2}\, d\tau$$

$$= \int_1^t \sqrt{4\tau^2 + 4\tau^4}\, d\tau$$

$$= \int_1^t \sqrt{4\tau^2(1+\tau^2)}\, d\tau$$

$$= \int_1^t 2\tau\sqrt{1+\tau^2}\, d\tau$$

$$= \frac{2}{3}(1+\tau^2)^{3/2}\Big|_1^t$$

$$= \frac{2}{3}(1+t^2)^{3/2} - \frac{2}{3}(2)^{3/2}, \text{ i.e.,}$$

arc length is

$$s(t) = \frac{2}{3}(1+t^2)^{3/2} - \frac{2}{3}(2)^{3/2}.$$

arc length from $t=1$ to $t=4$ is

$$s(4) = \frac{2}{3}(17)^{3/2} - \frac{2}{3}(2)^{3/2} \approx 44.84$$

<u>Example</u>: Consider path C in 3D-Space determined by the position vector

$$\vec{r}(t) = (2\sin 2t)\vec{i} + (2\cos 2t)\vec{j} + (3t)\vec{k}.$$

First find the arc length of C from $t=0$ to t. Then find the arc length from $t=0$ to $t=2$.

$$s(t) = \int_0^t \sqrt{(4\cos 2t)^2 + (-4\sin 2t)^2 + (3)^2}\, d\tau$$

$$= \int_0^t \sqrt{16\cos^2 2t + 16\sin^2 2t + 9}\, d\tau$$

$$= \int_0^t \sqrt{16(\cos^2 2t + \sin^2 2t) + 9}\, d\tau$$

$$= \int_0^t \sqrt{16(1) + 9}\, d\tau$$

$$= \int_0^t \sqrt{25}\, d\tau = \int_0^t 5\, d\tau$$

$$= 5\tau \Big|_0^t = 5t,$$ i.e., arc length is

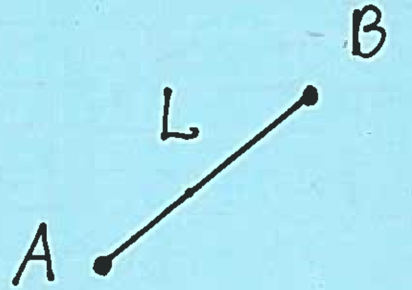
$$s(t) = 5t.$$

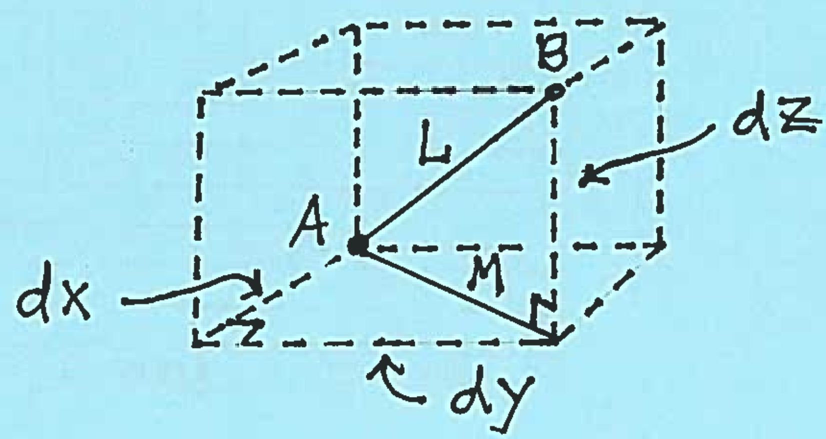
arc length from $t=0$ to $t=2$ is

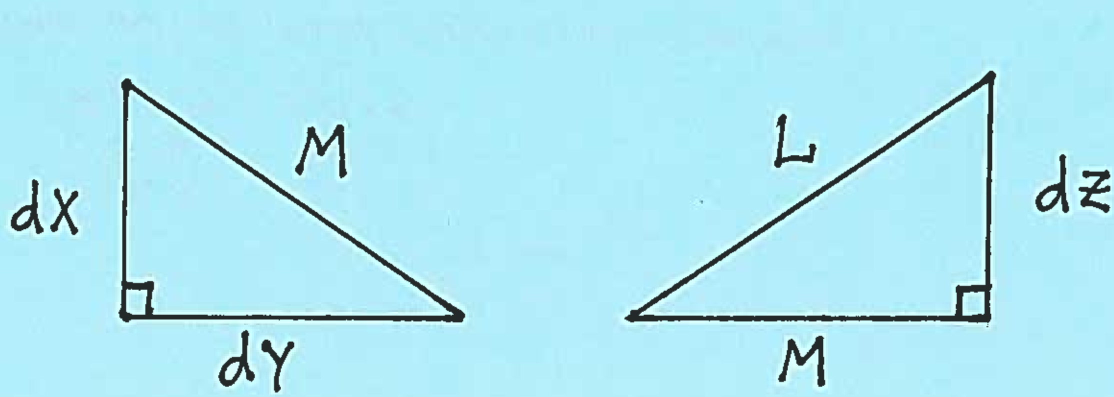
$$s(2) = 5(2) = 10$$

What is the origin of the definition of arc Length?

Consider finding the distance L between two points A and B in 3D-Space. Form a rectangular box of dimensions dx, dy, and dz and with L as its main diagonal:







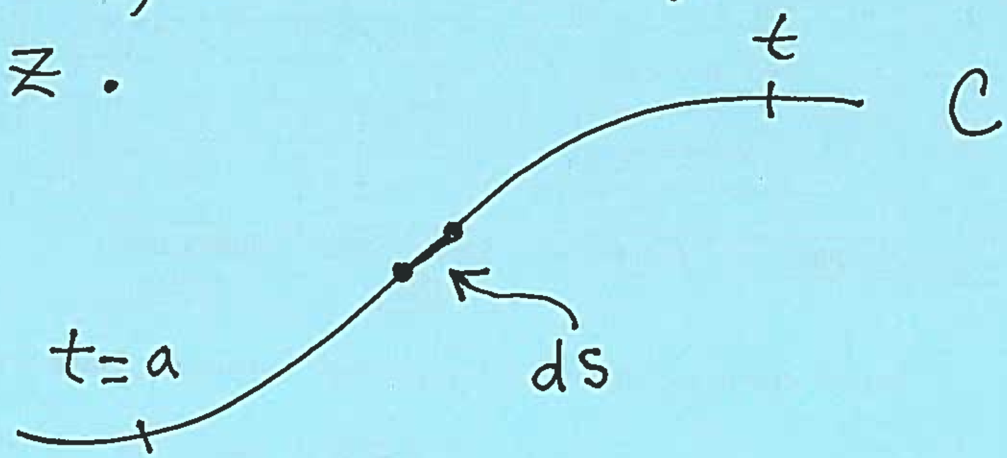
By the Pythagorea Theorem

$(dx)^2 + (dY)^2 = M^2$ and $M^2 + (dZ)^2 = L^2 \rightarrow$

$$L^2 = (dx)^2 + (dY)^2 + (dZ)^2 \rightarrow$$

$$L = \sqrt{(dx)^2 + (dY)^2 + (dZ)^2} \, .$$

Let's apply this idea to arc length on path C. Consider a small piece of arc length ds determined by dx, dY, and dZ, small changes in X, Y, and Z.



The arc length of C from t=a

to t is

$$s(t) = \int ds$$

$$= \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= \int_a^t \frac{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}{d\tau} \cdot d\tau$$

$$= \int_a^t \sqrt{\frac{(dx)^2}{(d\tau)^2} + \frac{(dy)^2}{(d\tau)^2} + \frac{(dz)^2}{(d\tau)^2}}\, d\tau$$

$$= \int_a^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau,$$

i.e., arc length is

$$s(t) = \int_a^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2}\, d\tau.$$