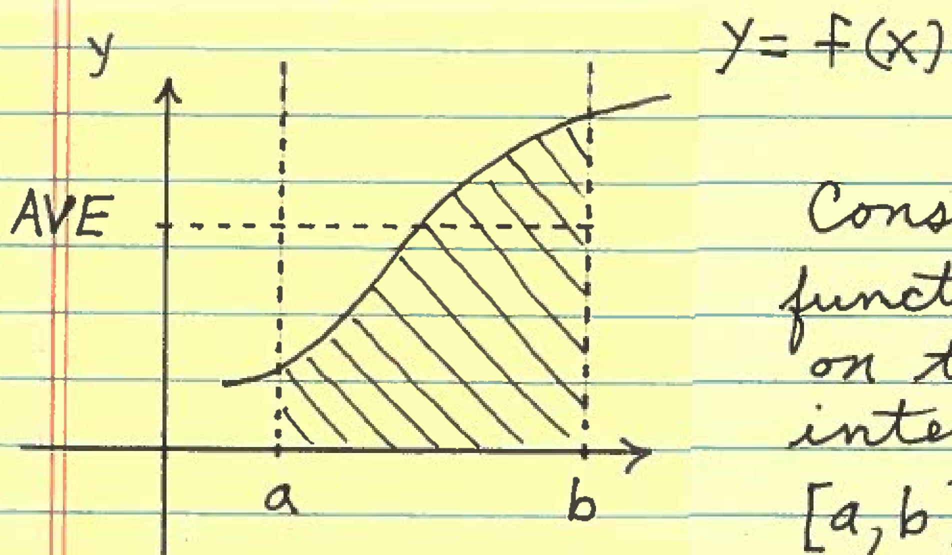
Section 15.2
Thomas Calculus
11th Ed.

Average Value of Function $z = f(x,y)$ on Region R in the xy-Plane

Recall: (from Math 21B)



Consider function $y = f(x)$ on the interval $[a,b]$, and

the shaded region below the graph. Now CREATE a rectangle of height AVE and whose base is the interval $[a,b]$, so that the

Area of Rectangle $\doteq$ Shaded area

$\rightarrow$ $(\text{base})(\text{height}) = \int_a^b f(x)\,dx$

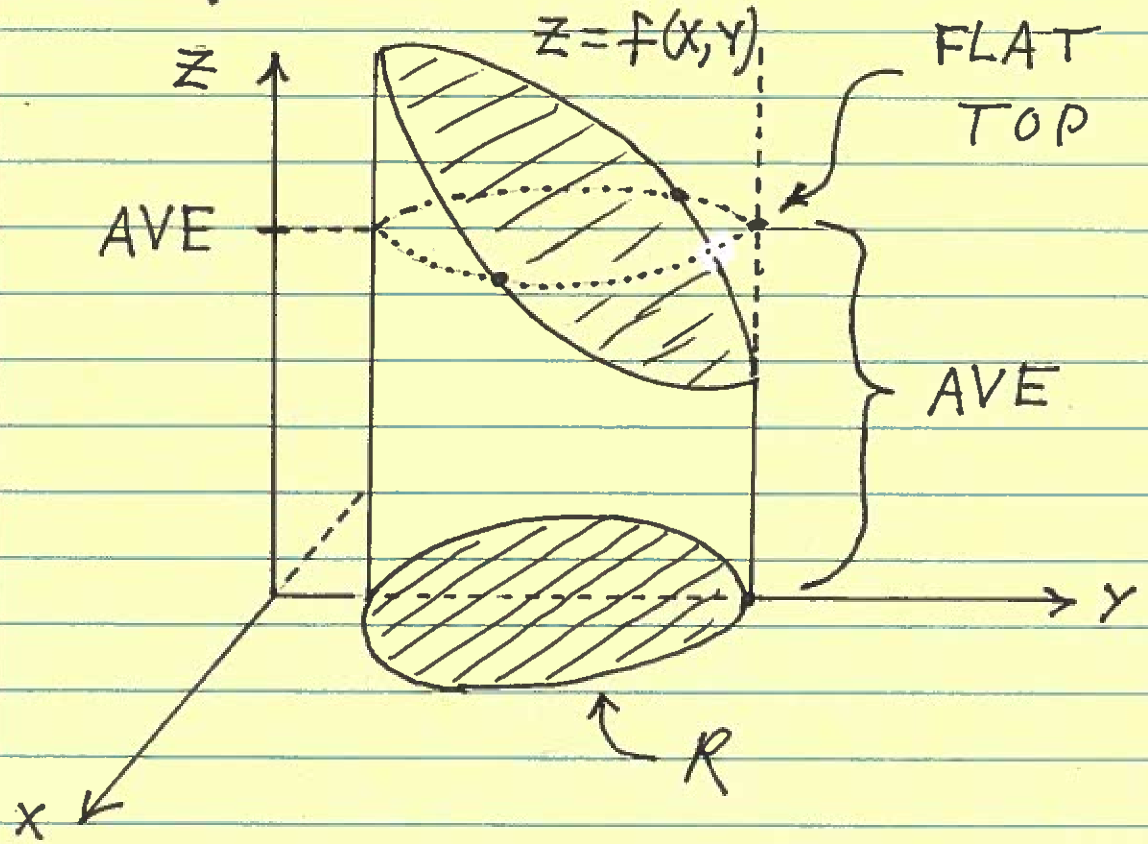
$\rightarrow$ $(b-a)(\text{AVE}) = \int_a^b f(x)\,dx$

$\rightarrow$ $$\boxed{\text{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

This is the Definition of the <u>average</u> <u>Value</u> of $y = f(x)$ on the interval $[a, b]$.

Let's generalize this process to create an analogous Definition for the average Value of $z = f(x, y)$ over region R in the xy-plane:

Consider the SOLID below a surface $z = f(x, y)$ and above a region R in the xy-plane:



Now create a new "cylinder" of height AVE and whose base is region R, so that the

Volume of "cylinder" = Volume of SOLID

$$\rightarrow (\text{base})(\text{height}) = \iint_R f(x,y)\, dA$$

$\rightarrow$ $(\text{area of } R)(AVE) = \iint\limits_R f(x,y)\, dA$

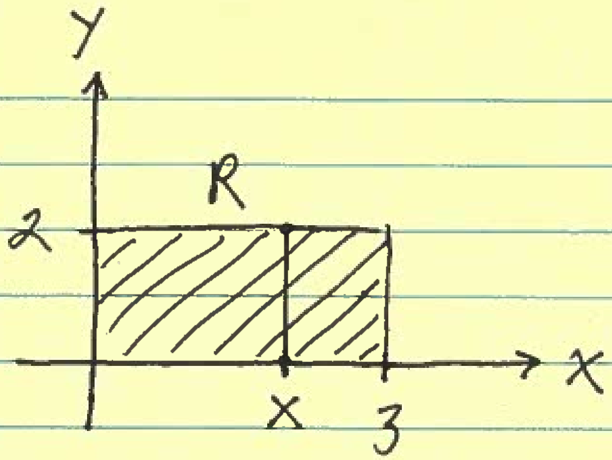
$$\rightarrow \quad \boxed{AVE = \frac{1}{\text{area } R} \iint\limits_R f(x,y)\, dA}$$

This is the Definition of the average Value of $z = f(x,y)$ over region R in the xy-plane.

Example: Find the average Value of $z = f(x,y)$ over region R, which is bounded by the graphs of the given equations.

1.) $z = 2xy$;

$R: y=0, x=0, y=2, x=3$



area $R = (3)(2)$

$= 6$,

and

$$R: \begin{cases} 0 \le x \le 3 \\ 0 \le y \le 2 \end{cases}, \text{ then}$$

$$AVE = \frac{1}{6} \int_0^3 \int_0^2 2xy \, dy \, dx$$

$$= \frac{1}{6} \int_0^3 (xy^2) \Big|_{y=0}^{y=2} dx$$

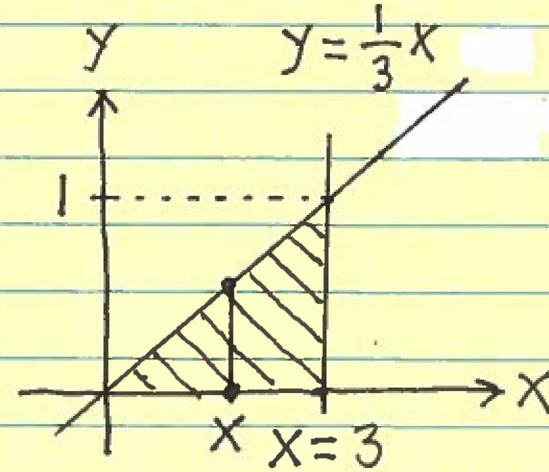
$$= \frac{1}{6} \int_0^3 4x \, dx$$

$$= \frac{1}{6} \cdot (2x^2) \Big|_0^3 = \frac{1}{6}(18) = \boxed{3}$$

2.) $z = 6x + 3y$;

$R: y = \frac{1}{3}x$, $x = 3$, $y = 0$

$$R: \begin{cases} 0 \le x \le 3 \\ 0 \le y \le \frac{1}{3}x \end{cases}$$



then area $R = \frac{1}{2}(\text{base})(\text{height})$

$$= \frac{1}{2}(3)(1) = \frac{3}{2}, \text{ and}$$

$$AVE = \frac{1}{\frac{3}{2}} \int_0^3 \int_0^{\frac{1}{3}x} (6x + 3y)\, dy\, dx$$

$$= \frac{2}{3} \int_0^3 \left[6xy + \frac{3}{2}y^2\right]\Big|_{y=0}^{y=\frac{1}{3}x} dx$$

$$= \frac{2}{3} \int_0^3 \left[\left(6x\left(\tfrac{1}{3}x\right) + \tfrac{3}{2}\left(\tfrac{1}{3}x\right)^2\right) - (0+0)\right] dx$$

$$= \frac{2}{3} \int_0^3 \left(2x^2 + \frac{1}{6}x^2\right) dx$$

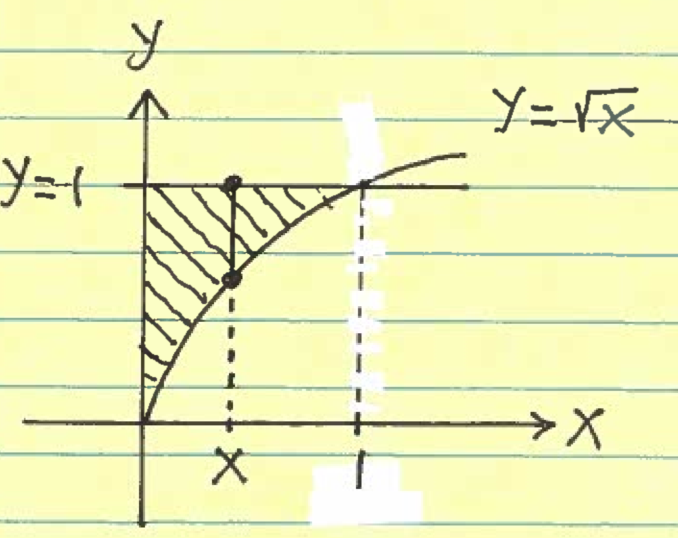
$$= \frac{2}{3} \cdot \int_0^3 \frac{13}{6} x^2\, dx$$

$$= \frac{2}{3} \cdot \frac{13}{6} \cdot \frac{1}{3} x^3 \Big|_0^3$$

$$= \frac{13}{9} \cdot (9) = 13$$

3.) $z = \dfrac{xy}{1+y^2}$;

$R: y = \sqrt{x},\ y = 1,\ x = 0$



$R: \begin{cases} 0 \le x \le 1 \\ \sqrt{x} \le y \le 1 \end{cases}$, ~~then~~

$$\text{area } R = \int_0^1 \int_{\sqrt{x}}^1 1 \, dy \, dx$$

$$= \int_0^1 y \Big|_{y=\sqrt{x}}^{y=1} dx = \int_0^1 (1-\sqrt{x})\, dx$$

$$= \left(x - \frac{2}{3}x^{3/2}\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \text{ ; and}$$

$$AVE = \frac{1}{\frac{1}{3}} \int_0^1 \int_{\sqrt{x}}^1 \frac{xy}{1+y^2}\, dy\, dx$$

$$= 3 \int_0^1 \int_{\sqrt{x}}^1 x \cdot \frac{y}{1+y^2}\, dy\, dx$$

$$= 3 \int_0^1 \left(x \cdot \frac{1}{2} \ln(1+y^2)\right)\Big|_{y=\sqrt{x}}^{y=1} dx$$

$$= 3 \int_0^1 \left[\frac{1}{2}x \ln 2 - \frac{1}{2}x \ln(1+x)\right] dx$$

$$= 3 \cdot \frac{1}{2} \int_0^1 x \ln 2 \, dx - \frac{3}{2} \int_0^1 x \ln(1+x)\, dx$$

$$= \frac{3}{2} \ln 2 \cdot \frac{1}{2} x^2 \Big|_0^1 - \frac{3}{2} \int_0^1 x \ln(1+x)\, dx$$

$$= \frac{3}{4}\ln 2 - \frac{3}{2}\int_0^1 x\ln(1+x)\,dx$$

(Let $u = \ln(1+x)$, $dv = x\,dx$

$\rightarrow du = \frac{1}{1+x}dx$, $v = \frac{1}{2}x^2$), so

$$\int_0^1 x\ln(1+x)\,dx$$

$$= \frac{1}{2}x^2\ln(1+x)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x}\,dx$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1 \left(x - 1 + \frac{1}{1+x}\right)dx$$

$$\begin{array}{r|l} & x-1 \\ \hline 1+x & x^2 \\ & -(x^2+x) \\ \hline & -x \\ & -(-x-1) \\ \hline & 1 \end{array}$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\left\{\left(\frac{1}{2}x^2 - x + \ln|1+x|\right)\Big|_0^1\right\}$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\left\{\left(\frac{1}{2} - 1 + \ln 2\right) - \left(0 - 0 + \cancel{\ln 1}\right)\right\}$$

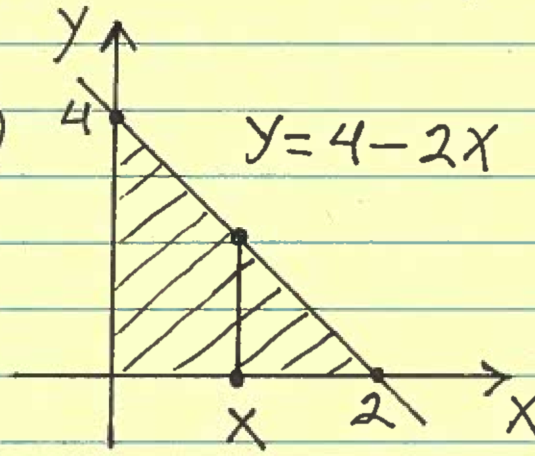
$$= \frac{1}{2}\cancel{\ln 2} + \frac{1}{4} - \frac{1}{2}\cancel{\ln 2} = \frac{1}{4}, \text{ then}$$

$$\text{AVE} = \frac{3}{4}\ln 2 - \frac{3}{2}\left\{\frac{1}{4}\right\} = \frac{3}{4}\ln 2 - \frac{3}{8}$$

Example: a metal plate lies in the region bounded by the graphs of $y=4-2x$, $y=0$, and $x=0$. The temperature at point (x,y) is given by

$$T = x - y + 4 \quad (°F).$$

Find the Average Temperature of the plate. (SET UP ONLY)



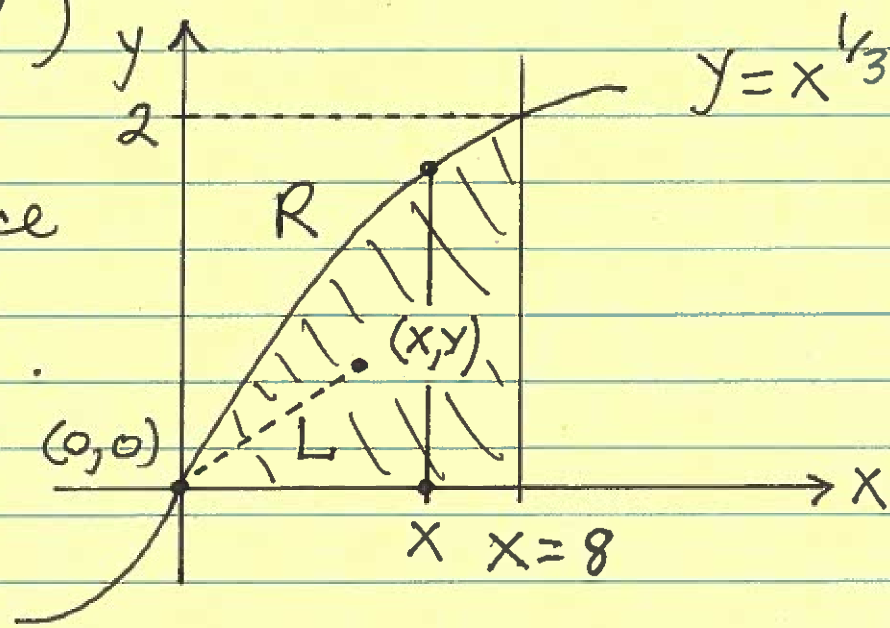
Area plate $= \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(2)(4) = 4$, and

$$\text{AVE} = \frac{1}{4}\int_0^2 \int_0^{4-2x} (x-y+4)\,dy\,dx \quad (°F)$$

(NOTE: If you work it out, the answer is $\frac{10}{3}$ °F.)

<u>Example</u>: Find the Average Distance between the origin $(0,0)$ and all points (x,y) in the region R, which is bounded by the graphs of $y = x^{1/3}$, $y = 0$, and $x = 8$. (SET UP ONLY)



The Distance function between points $(0,0)$ and (x,y) is

$$L = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2};$$

and

$$R: \begin{cases} 0 \le x \le 8 \\ 0 \le y \le x^{1/3} \end{cases} \longrightarrow$$

Area $R = \int_0^8 \int_0^{x^{1/3}} 1 \, dy \, dx$ and

the average Distance is

$$AVE = \frac{1}{\int_0^8 \int_0^{x^{1/3}} 1 \, dy \, dx} \int_0^8 \int_0^{x^{1/3}} \sqrt{x^2+y^2} \, dy \, dx$$