Example: A thin wire lies along the graph of $y = \sqrt{4 - x^2}$ in the first quadrant. Density at point $P = (x, y)$ is given by

$$\delta(P) = \delta(x, y) = x + 3y^2 \quad \text{gm./cm.}$$

Find the wire's

1.) Total Mass.

2.) Centroid.

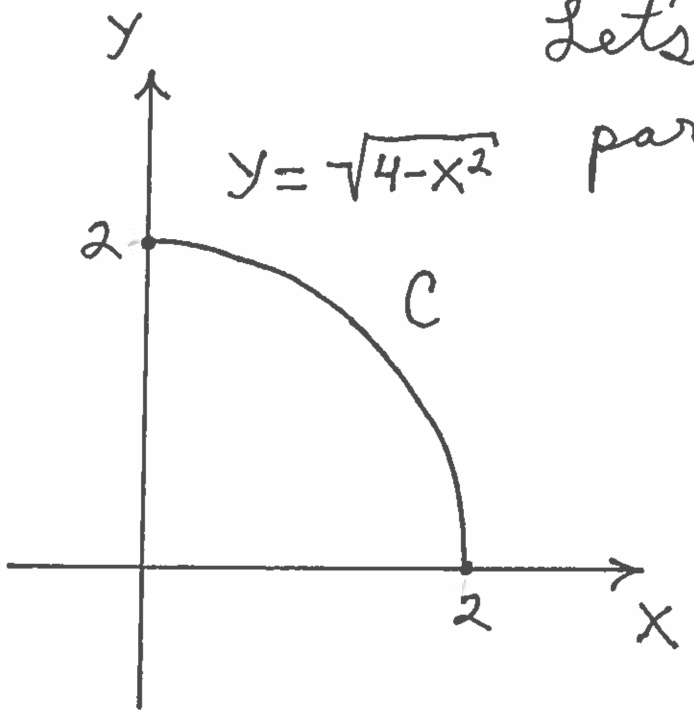
3.) Center of Mass.

4.) Moment of Inertia about the

a.) origin.

b.) line $x = 3$.

c.) x-axis.



Let's first parameterize path C :

$$C : \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$

for $0 \le t \le \frac{\pi}{2}$

or $C : \vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} \xrightarrow{D}$

$\vec{v}(t) = (-2\sin t)\vec{i} + (2\cos t)\vec{j} \longrightarrow$

$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{4(1)} = 2$$; Now

density is

$$\delta(P) = 2\cos t + 3(2\sin t)^2$$

$= 2\cos t + 12\sin^2 t$; then

1.) $\text{Mass} = \int_C \delta(P)\, ds = \int_C \delta(P) \frac{ds}{dt}\, dt$

$= \int_0^{\frac{\pi}{2}} (2\cos t + 12\sin^2 t)(2)\, dt$

$= \int_0^{\frac{\pi}{2}} (4\cos t + 24\sin^2 t)\, dt$

$= \int_0^{\frac{\pi}{2}} \left(4\cos t + 24 \cdot \frac{1}{2}(1 - \cos 2t)\right) dt$

$= \int_0^{\frac{\pi}{2}} (4\cos t + 12 - 12\cos 2t)\, dt$

$= \left(4\sin t + 12t - 6\sin 2t\right)\Big|_0^{\frac{\pi}{2}}$

$= \left(4\sin\left(\frac{\pi}{2}\right) + 12\left(\frac{\pi}{2}\right) - 6\overset{0}{\cancel{\sin(\pi)}}\right)$

$- \left(4\overset{0}{\cancel{\sin(0)}} + \overset{0}{\cancel{12(0)}} - 6\overset{0}{\cancel{\sin(0)}}\right)$

$= 4(1) + 6\pi$

$= 4 + 6\pi \approx 22.85$ gm.

2.) For Centroid:

$$\bar{x} = \frac{\int_C x\,ds}{\int_C 1\,ds} \;;\quad \int_C 1\,ds = \int_0^{\frac{\pi}{2}} \frac{ds}{dt}\,dt$$

$$= \int_0^{\frac{\pi}{2}} 2\,dt = 2t\Big|_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2}\right) = \pi \text{ cm} \;;$$

$$\int_C x\,ds = \int_0^{\frac{\pi}{2}} x\,\frac{ds}{dt}\,dt = \int_0^{\frac{\pi}{2}} (2\cos t)(2)\,dt$$

$$= \int_0^{\frac{\pi}{2}} 4\cos t\,dt = 4\sin t\Big|_0^{\frac{\pi}{2}}$$

$$= 4\sin\left(\frac{\pi}{2}\right) - 4\sin(0)$$

$$= 4(1) - 4(0) = 4 \text{ cm}^2, \text{ so}$$

$$\bar{x} = \frac{4}{\pi} \approx 1.27 \text{ cm.} \;; \text{ and}$$

$$\bar{y} = \frac{\int_C y\,ds}{\int_C 1\,ds} \;;\quad \int_C y\,ds = \int_0^{\frac{\pi}{2}} y\,\frac{ds}{dt}\,dt$$

$$= \int_0^{\frac{\pi}{2}} (2\sin t)(2)\,dt = \int_0^{\frac{\pi}{2}} 4\sin t\,dt$$

$$= -4\cos t \Big|_0^{\frac{\pi}{2}} = -4\cos\left(\tfrac{\pi}{2}\right) - -4\cos(0)$$

$$= -4(0) + 4(1) = 4 \text{ cm.}^2; \text{ so}$$

$$\bar{y} = \frac{4}{\pi} \approx 1.27 \text{ cm.}$$

3.) For Center of Mass:

$$\bar{X} = \frac{\int_C x\,\delta(P)\,ds}{\int_C \delta(P)\,ds}; \quad \int_C x\,\delta(P)\,ds = \int_0^{\frac{\pi}{2}} x\,\delta(P)\,\frac{ds}{dt}\,dt$$

$$= \int_0^{\frac{\pi}{2}} (2\cos t)\cdot(2\cos t + 12\sin^2 t)(2)\,dt$$

$$= \int_0^{\frac{\pi}{2}} (8\cos^2 t + 48\cos t\sin^2 t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left(8\cdot\tfrac{1}{2}(1+\cos 2t) + 48\cos t\sin^2 t\right)dt$$

$$= \int_0^{\frac{\pi}{2}} \left(4 + 4\cos 2t + 48 \cos t \sin^2 t\right) dt$$

$$= \left(4t + 2\sin 2t + 16 \sin^3 t\right) \Big|_0^{\frac{\pi}{2}}$$

$$= \left(4\left(\tfrac{\pi}{2}\right) + 2\overset{0}{\cancel{\sin(\pi)}} + 16 \sin^3\left(\tfrac{\pi}{2}\right)\right)$$
$$- \left(4(0) + 2\overset{0}{\cancel{\sin(0)}} + 16 \overset{0}{\cancel{\sin^3(0)}}\right)$$

$$= 2\pi + 16 \approx 22.28 \ (\text{gm.})(\text{cm.}), \text{ so}$$

$$\bar{x} = \frac{2\pi + 16}{4 + 6\pi} \approx 0.98 \text{ cm. }; \text{ and}$$

$$\bar{y} = \frac{\int_C y\,\delta(P)\,ds}{\int_C \delta(P)\,ds}; \quad \int_C y\,\delta(P)\,ds = \int_0^{\frac{\pi}{2}} y\,\delta(P)\,\frac{ds}{dt}\,dt$$

$$= \int_0^{\frac{\pi}{2}} (2\sin t)(2\cos t + 12\sin^2 t)(2)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left(8 \sin t \cos t + 48 \sin^3 t\right) dt$$

$$= \int_0^{\frac{\pi}{2}} \left(8 \sin t \cos t + 48 \sin t \cdot \sin^2 t\right) dt$$

$$= \int_0^{\frac{\pi}{2}} (8\sin t\cos t + 48\sin t(1-\cos^2 t))\,dt$$

$$= \int_0^{\frac{\pi}{2}} (8\sin t\cos t + 48\sin t - 48\sin t\cos^2 t)\,dt$$

$$= \left(8\cdot\tfrac{1}{2}\sin^2 t - 48\cos t - 48\cdot\tfrac{-1}{3}\cos^3 t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(4\sin^2\left(\tfrac{\pi}{2}\right) - 48\overset{0}{\cancel{\cos}}\left(\tfrac{\pi}{2}\right) + 16\overset{0}{\cancel{\cos}}{}^3\left(\tfrac{\pi}{2}\right)\right)$$

$$- \left(4\overset{0}{\cancel{\sin}}{}^2(0) - 48\cos(0) + 16\cos^3(0)\right)$$

$$= (4(1) - 0) - (-48(1) + 16(1))$$

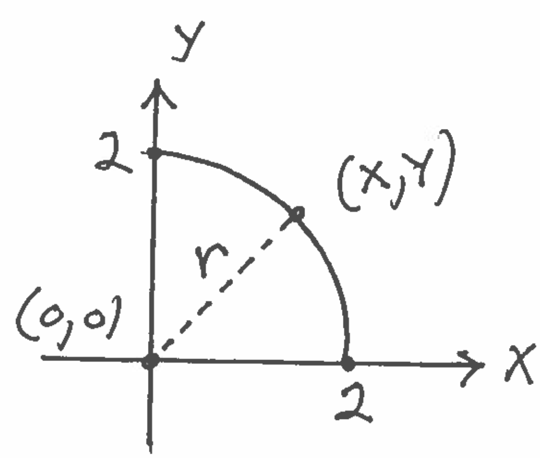
$$= 4 + 48 - 16 = 32, \text{ so}$$

$$\bar{y} = \frac{32}{4+6\pi} \approx 1.4 \text{ cm.}$$

4.) Moment of Inertia :

$$\text{M. of I.} = \int_C (\text{distance})^2\,\delta(P)\,ds$$

a.) about the origin:



so distance from (x,y) to $(0,0)$ is distance $= r = 2$, then

$$\text{M. of I.} = \int_C (2)^2 \delta(P)\, ds = \int_0^{\frac{\pi}{2}} 4\, \delta(P) \frac{ds}{dt}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 4(2\cos t + 12\sin^2 t)(2)\, dt$$

$$= \int_0^{\frac{\pi}{2}} (16\cos t + 96\sin^2 t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left(16\cos t + 96 \cdot \tfrac{1}{2}(1 - \cos 2t)\right) dt$$

$$= \int_0^{\frac{\pi}{2}} (16\cos t + 48 - 48\cos 2t)\, dt$$

$$= \left(16\sin t + 48t - 48 \cdot \tfrac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}}$$

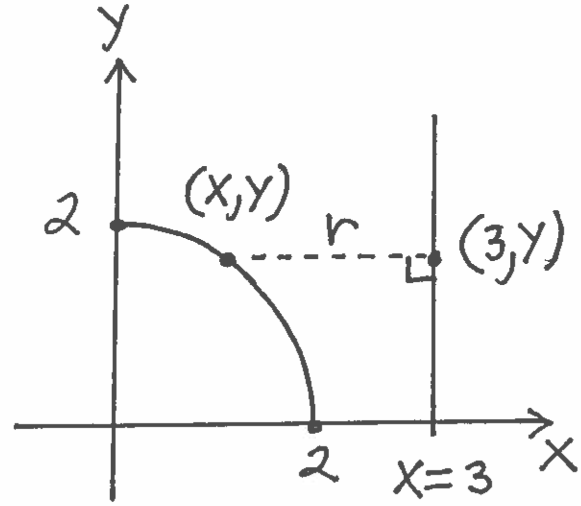
$$= \left(16\sin\left(\tfrac{\pi}{2}\right) + 48\left(\tfrac{\pi}{2}\right) - 24\,\overset{0}{\cancel{\sin(\pi)}}\right)$$

$$- \left(16\,\overset{0}{\cancel{\sin(0)}} + \overset{0}{\cancel{48(0)}} - 24\,\overset{0}{\cancel{\sin(0)}}\right)$$

$$= 16(1) + 24\pi$$

$$= 16 + 24\pi \approx 91.4 \quad (\text{gm.})(\text{cm.}^2)$$

b.) about the line $X=3$:



so distance from (X,Y) to $(3,Y)$ is

$$\text{distance} = r = \sqrt{(X-3)^2+(Y-Y)^2}$$

$$= \sqrt{(X-3)^2} = |X-3| = 3-X, \text{ then}$$

$$M.\text{ of }I. = \int_C (3-X)^2\,\delta(P)\cdot ds$$

$$= \int_0^{\frac{\pi}{2}} (X^2-6X+9)\cdot \delta(P)\cdot \frac{ds}{dt}\cdot dt$$

$$= \int_0^{\frac{\pi}{2}} \left((2\cos t)^2-6(2\cos t)+9\right)(2\cos t+12\sin^2 t)(2)\,dt$$

$$= \int_0^{\frac{\pi}{2}} (4\cos^2 t-12\cos t+9)(4\cos t+24\sin^2 t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} [16\cos^3 t+96\sin^2 t\cos^2 t-48\cos^2 t$$

$$-288\cos t\sin^2 t+36\cos t+216\sin^2 t]\,dt$$

(Sorry, but the solution gets "messy" now. I will use the following Trig Identities.

I.) $\cos^2 t = \frac{1}{2}(1+\cos 2t)$

II.) $\sin^2 t = \frac{1}{2}(1-\cos 2t)$

III.) $\sin^2 t \cos^2 t = (\sin t \cos t)^2$

$$= \left(\frac{1}{2}\cdot 2 \sin t \cos t\right)^2$$

$$= \left(\frac{1}{2}\sin 2t\right)^2$$

$$= \frac{1}{4}\sin^2 2t$$

$$= \frac{1}{4}\cdot\frac{1}{2}(1-\cos 4t)$$

$$= \frac{1}{8}(1-\cos 4t)$$

IV.) $\cos^3 t = \cos t \cdot \cos^2 t$

$$= \cos t \cdot (1-\sin^2 t)$$

$$= \cos t - \sin^2 t \cos t \quad .)$$

$$= \int_0^{\frac{\pi}{2}} \Big\{ 16(\cos t - \sin^2 t \cos t)$$
$$+ 96 \cdot \frac{1}{8}(1 - \cos 4t)$$
$$- 48 \cdot \frac{1}{2}(1 + \cos 2t)$$
$$- 288 \cos t \sin^2 t$$
$$+ 36 \cos t$$
$$+ 216 \cdot \frac{1}{2}(1 - \cos 2t) \Big\} \, dt$$

$$= \int_0^{\frac{\pi}{2}} \Big\{ 16 \cos t - 16 \sin^2 t \cos t$$
$$+ 12 - 12 \cos 4t - 24 - 24 \cos 2t$$
$$- 288 \cos t \cdot \sin^2 t + 36 \cos t$$
$$+ 108 - 108 \cos 2t \Big\} \, dt$$

$$= \int_0^{\frac{\pi}{2}} \Big\{ 96 + 52 \cos t - 132 \cos 2t$$
$$- 12 \cos 4t - 304 \cos t \sin^2 t \Big\} \, dt$$

$$= \left[96t + 52\sin t - 132\cdot\tfrac{1}{2}\sin 2t - 12\cdot\tfrac{1}{4}\sin 4t - 304\cdot\tfrac{1}{3}\sin^3 t\right]\Bigg|_0^{\frac{\pi}{2}}$$

$$= \left[96\left(\tfrac{\pi}{2}\right) + 52\sin\left(\tfrac{\pi}{2}\right) - 66\,\cancelto{0}{\sin}(\pi) - 3\,\cancelto{0}{\sin}(2\pi) - \frac{304}{3}\sin^3\left(\tfrac{\pi}{2}\right)\right]$$

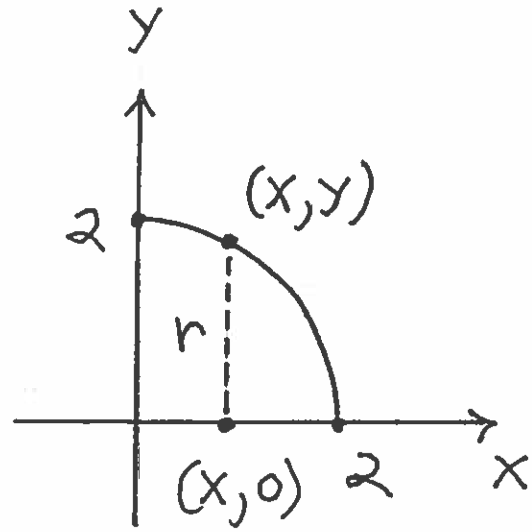
$$- \left[\cancelto{0}{96(0)} + 52\,\cancelto{0}{\sin}(0) - 66\,\cancelto{0}{\sin}(0) - 3\,\cancelto{0}{\sin}(0) - \frac{304}{3}\,\cancelto{0}{\sin^3}(0)\right]$$

$$= 48\pi + 52(1) - \frac{304}{3}(1)$$

$$= 48\pi + \frac{156}{3} - \frac{304}{3}$$

$$= 48\pi - \frac{148}{3} \approx 101.46\ (\text{gm.})(\text{cm.}^2)$$

c.) about the x-axis: so distance from (x,y) to $(x,0)$ is



$$\text{distance} = r = \sqrt{(x-x)^2+(y-0)^2}$$

$$= \sqrt{y^2} = |y| = y \text{ , then}$$

$$\text{M. of I.} = \int_C (y)^2 \delta(P)\, ds = \int_0^{\frac{\pi}{2}} y^2 \cdot \delta(P) \frac{ds}{dt}\, dt$$

$$= \int_0^{\frac{\pi}{2}} (2\sin t)^2 (2\cos t + 12\sin^2 t)(2)\, dt$$

$$= \int_0^{\frac{\pi}{2}} (16\sin^2 t\cos t + 96\sin^4 t)\, dt$$

(Trig Identity:

$$\sin^4 t = (\sin^2 t)^2$$

$$= \left(\tfrac{1}{2}(1-\cos 2t)\right)^2$$

$$= \frac{1}{4}\left(1 - 2\cos 2t + \cos^2 2t\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{4}\cdot\frac{1}{2}\left(1 + \cos 4t\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{8} + \frac{1}{8}\cos 4t$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t)$$

$$= \int_0^{\frac{\pi}{2}} \Big(16\sin^2 t\cos t + 96\left(\frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t\right)\Big)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left(16\sin^2 t\cos t + 36 - 48\cos 2t + 12\cos 4t\right)dt$$

$$= \left(16\cdot\frac{1}{3}\sin^3 t + 36t - 48\cdot\frac{1}{2}\sin 2t + 12\cdot\frac{1}{4}\sin 4t\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left[\frac{16}{3}\sin^3\left(\frac{\pi}{2}\right) + 36\left(\frac{\pi}{2}\right) - 24\,\overset{0}{\cancel{\sin}(\pi)} + 3\,\overset{0}{\cancel{\sin}(2\pi)}\right]$$

$$- \left[\frac{16}{3}\,\overset{0}{\cancel{\sin^3}(0)} + \overset{0}{\cancel{36}(0)} - 24\,\overset{0}{\cancel{\sin}(0)} + 3\,\overset{0}{\cancel{\sin}(0)}\right]$$

$$= \frac{16}{3}(1) + 18\pi = \frac{16}{3} + 18\pi \approx 16.88\ (\text{gm.})(\text{cm.}^2)$$