Section 15.7
Thomas Calculus
11th Ed.

Change of Variables, Jacobians

Recall : I.) 2X2 Determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

II.) 3X3 Determinant is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example : 1.) $\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = (2)(4) - (-3)(1)$
$= 11$

2.) $$\begin{vmatrix} 1 & -2 & 0 \\ 3 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} = (1)\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}$$

$$-(-2)\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} + (0)\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$$

$= ((0)(1) - (-1)(1)) + 2((3)(1) - (-1)(2)) + 0$

$= 1 + 2(5) = 11$

__Definition__: Consider the plane-to-plane mapping

$$F(u,v) = (f(u,v), g(u,v)) = (x,y).$$

The __Jacobian of F__ at the point $P = (u_0, v_0)$ is the 2×2 determinant

$$J(P) = \begin{vmatrix} f_u(u_0,v_0) & f_v(u_0,v_0) \\ g_u(u_0,v_0) & g_v(u_0,v_0) \end{vmatrix}.$$

__Example__: Let $F(u,v) = (u^2 v, u - v)$, then

$$J(P) = \begin{vmatrix} 2uv & u^2 \\ 1 & -1 \end{vmatrix}$$

$$= (2uv)(-1) - (u^2)(1) = -2uv - u^2;$$

and $J(-3,1) = -2(-3)(1) - (-3)^2$
$= 6 - 9 = -3$;

$J(1,-1) = -2(1)(-1) - (1)^2 = 1$;

$J(4,-2) = -2(4)(-2) - (4)^2 = 0$

<u>Definition</u>: If a plane-to-plane mapping is of the form

$$F(u,v) = (au+bv, cu+dv),$$

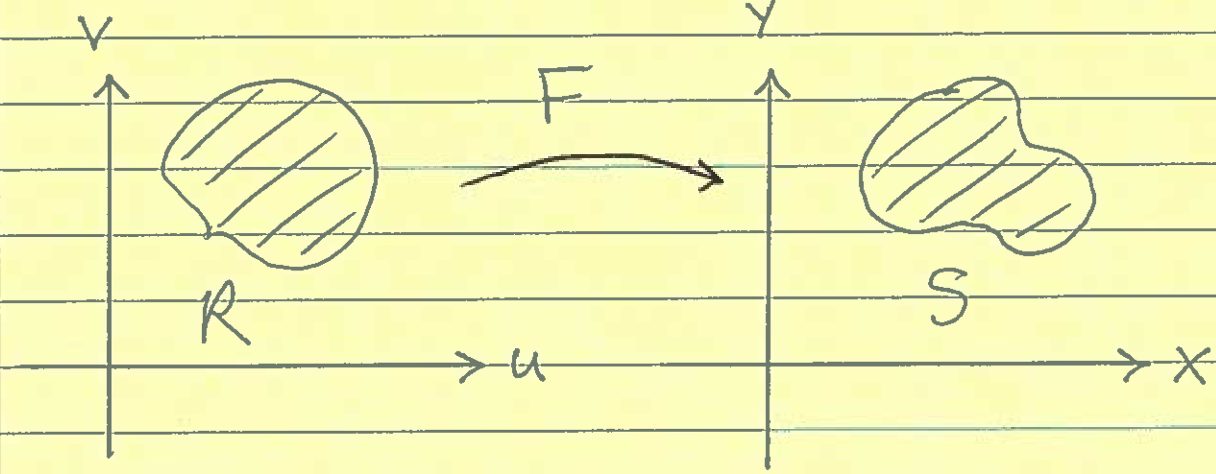
where $a, b, c,$ and d are constants, then F is called a <u>linear</u> <u>map</u>.

<u>Remarks</u>: 1.) F maps lines to lines.

2.) $J(P) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

<u>Theorem</u>: Let F be a linear map from plane to plane, let R be a closed region in the uv-plane, and let S

be the image of R under F in the xy-plane



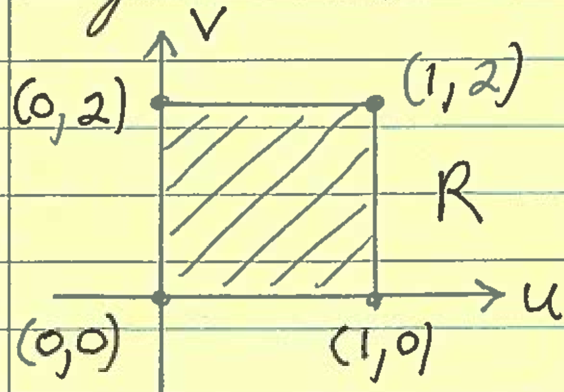
Then the Area of S is

$$\text{Area of } S = |ad - bc| \cdot \text{Area of } R \,.$$

Example: Let $F(u,v) = (u+v, 3u-v)$,

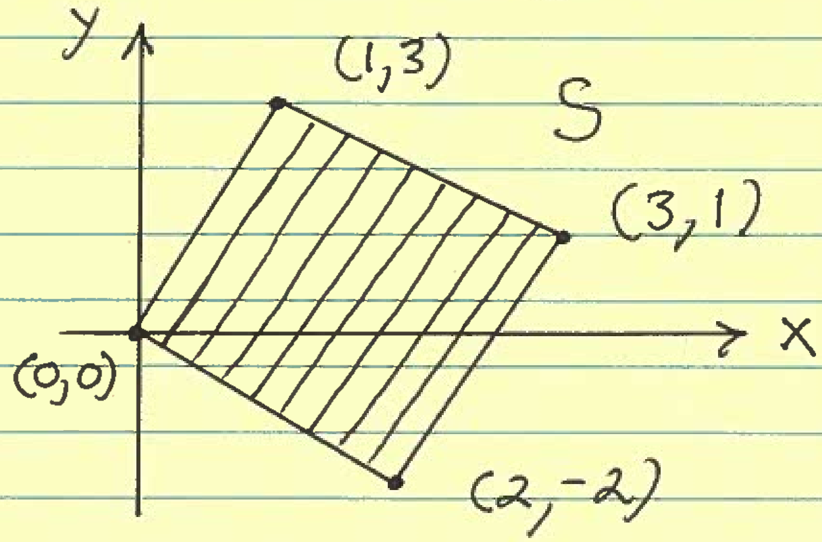
then $J(P) = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1-3 = -4$.

Consider the given rectangular region R:



$F(0,0) = (0,0)$,
$F(1,0) = (1,3)$,
$F(0,2) = (2,-2)$,
$F(1,2) = (3,1)$;

Let S be the image of R under F.



Then

$$\text{Area of } S = |(-1)(-1) - (1)(3)| \cdot \text{Area of } R$$

$$= |-4| \cdot 2 = 8$$

<u>Definition</u>: We call $|J(P_0)| = |ad - bc|$ the <u>magnification</u> of F at P_0.

<u>Question</u>: What if F is not a linear map? How do we compute magnification?

Answer: For $F(u,v) = (f(u,v), g(u,v))$

each of $f(u,v)$ and $g(u,v)$, each of which represents a surface in 3D-Space, is approximated with a tangent plane. This leads to the following theorem.

Theorem: Let $F(u,v) = (f(u,v), g(u,v))$

be a plane-to-plane mapping and let $P_0 = (u_0, v_0)$. Then the magnification of F at P_0 is given by

$$|J(P_0)| = \begin{vmatrix} f_u(P_0) & f_v(P_0) \\ g_u(P_0) & g_v(P_0) \end{vmatrix}.$$

Change of Variable Theorem:

Suppose we have a plane-to-plane mapping

$$F(u,v) = (f(u,v), g(u,v)) = (x,y)$$

defined on region R in the uv-plane, and with image S in the xy-plane.

If $F(P) = Q$, then

$$\iint_S h(Q)\, dA = \iint_S h(Q)\, dy\, dx$$

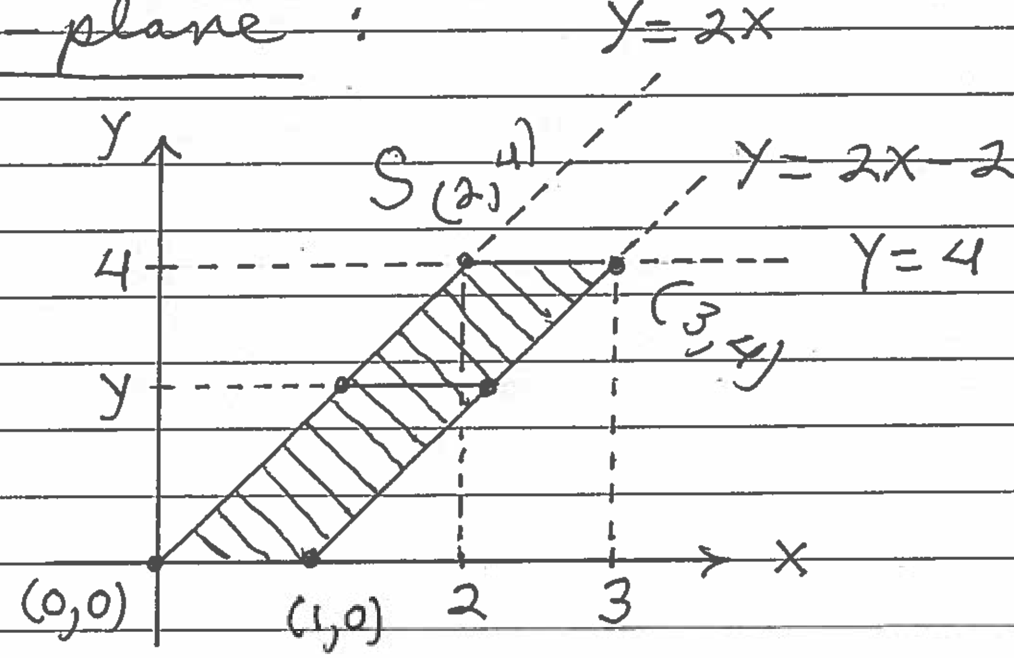
$$= \iint_R h(F(P)) \cdot |J(P)| \cdot dv\, du \quad .$$

Example: apply the previous theorem to each Double Integral and given change of variables.

1.) $\displaystyle\int_0^4 \int_{y/2}^{\frac{y}{2}+1} \left(x - \frac{y}{2}\right)\, dx\, dy$

with $u = x - \frac{1}{2}y$, $v = \frac{1}{2}y$;

First sketch region S in the xy-plane :



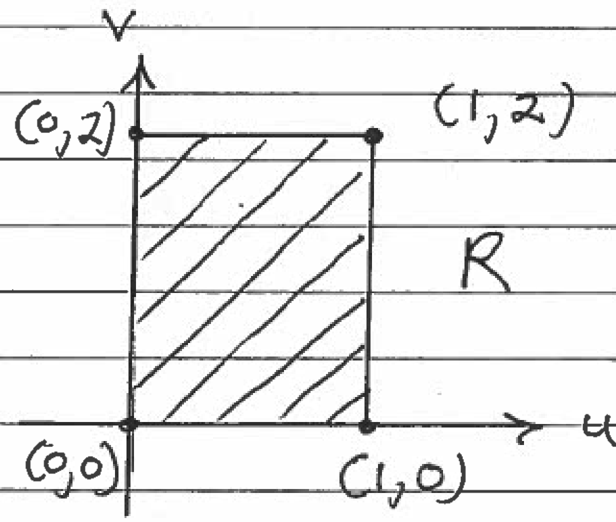
$$S: \begin{cases} 0 \le y \le 4 \\ \frac{y}{2} \le x \le \frac{y}{2} + 1 \end{cases}$$

Define function G to be

$$G(x,y) = \left(x - \tfrac{1}{2}y,\ \tfrac{1}{2}y\right) = (u,v) .$$

Map the corners of region S to the uv-plane and use the fact that G is linear, so G maps lines to lines:

$G(0,0) = (0,0)$,
$G(1,0) = (1,0)$,
$G(2,4) = (0,2)$,
$G(3,4) = (1,2)$



(edge $y = 2x$; $0 \le x \le 2$)
$G(x, 2x) = (x - x, x) = (0, x)$;
(edge $y = 2x - 2$; $1 \le x \le 3$)
$G(x, 2x-2) = (x - (x-1), x-1) = (1, x-1)$;
(edge $y = 4$; $2 \le x \le 3$)
$G(x, 4) = (x - 2, 2)$;
(edge $y = 0$; $0 \le x \le 1$)
$G(x, 0) = (x, 0)$.

Now write x and y in terms of u and v and define a new function F:

$$\begin{cases} u = x - \frac{1}{2}y \\ v = \frac{1}{2}y \end{cases} \longrightarrow$$

$$\boxed{u+v = x} \quad \text{and} \quad \boxed{2v = y}.$$

Define function F by

$$\boxed{F(u,v) = (u+v, 2v) = (x,y)}.$$

Describe region R, find the Jacobian, and transform the given Double Integral using the Change of Variable Theorem;

$$R: \begin{cases} 0 \le u \le 1 \\ 0 \le v \le 2 \end{cases}$$

The Jacobian is

$$J(P) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2 \text{ ; then}$$

$$\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \left(x - \tfrac{1}{2}y\right) dx\, dy$$

$$= \int_0^1 \int_0^2 [(u+v) - v]\, |J(P)|\, dv\, du$$

$$= \int_0^1 \int_0^2 [u](2)\, dv\, du$$

$$= \int_0^1 2uv \Big|_{v=0}^{v=2} du$$

$$= \int_0^1 (4u - 0)\, du$$

$$= 2u^2 \Big|_0^1$$

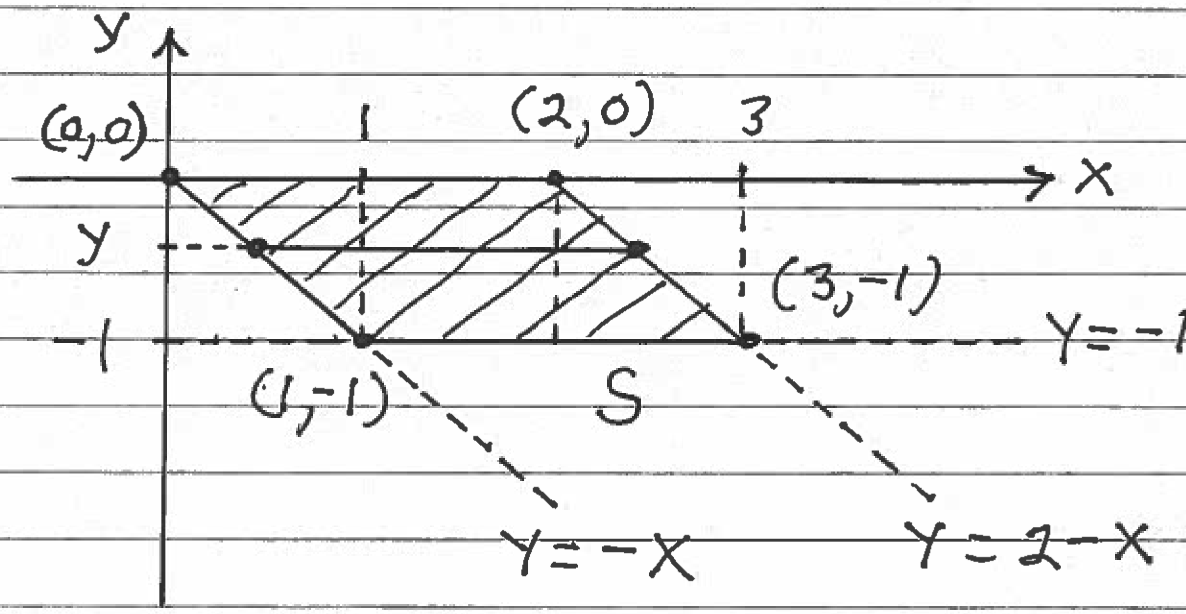
$$= 2 - 0$$

$$= 2 .$$

2.) $\int_{-1}^{0} \int_{-y}^{2-y} (x+y)^2 \, dx \, dy$

with $u = -y$, $v = \frac{1}{2}x + \frac{1}{2}y$;

First sketch region S in the xy-plane :

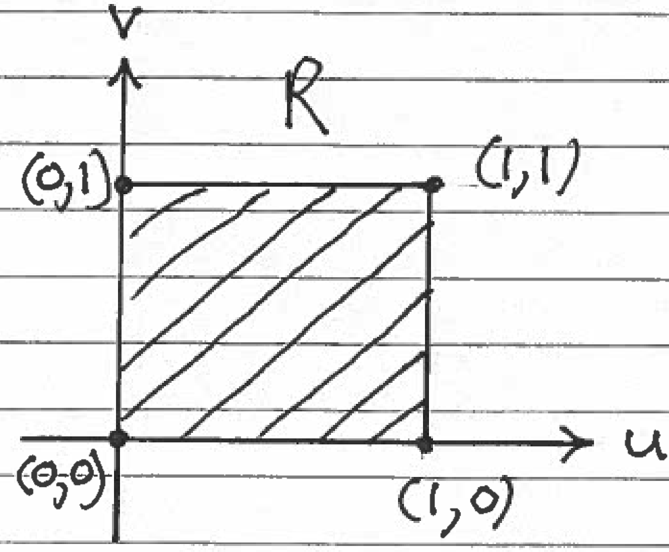


$$S : \begin{cases} -1 \le y \le 0 \\ -y \le x \le 2-y \end{cases}$$

Define function G to be

$$G(x,y) = \left(-y, \frac{1}{2}x + \frac{1}{2}y\right) = (u,v) .$$

Map the corners of region S to the uv-plane and use the fact that G is linear, so G maps lines to lines:



$G(0,0) = (0,0)$,
$G(2,0) = (0,1)$,
$G(1,-1) = (1,0)$,
$G(3,-1) = (1,1)$

(edge $y = -x$; $0 \le x \le 1$)
$G(x,-x) = (x,0)$;
(edge $y = -1$; $1 \le x \le 3$)
$G(x,-1) = (1, \frac{1}{2}x - \frac{1}{2})$;
(edge $y = 2-x$; $2 \le x \le 3$)
$G(x, 2-x) = (x-2, \frac{1}{2}x + \frac{1}{2}(2-x)) = (x-2, 1)$;
(edge $y = 0$; $0 \le x \le 2$)
$G(x,0) = (0, \frac{1}{2}x)$.

Now write x and y in terms of u and v and define a new function F:

$$\begin{cases} u = -y \\ v = \frac{1}{2}x + \frac{1}{2}y \end{cases} \longrightarrow \frac{1}{2}u = -\frac{1}{2}y$$

$$\rightarrow \frac{1}{2}u + v = \frac{1}{2}x \rightarrow$$

$$\boxed{u + 2v = x} \text{ and } \boxed{-u = y}.$$

Define function F by

$$\boxed{F(u,v) = (u+2v, -u) = (x,y)}.$$

Describe region R, find the Jacobian, and transform the given Double Integral using the Change of Variable Theorem:

$$R: \begin{cases} 0 \le u \le 1 \\ 0 \le v \le 1 \end{cases}$$

The Jacobian is

$$J(P) = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 0 - (-2) = 2 \;;$$

Then

$$\int_{-1}^{0} \int_{-y}^{2-y} (x+y)^2 \, dx \, dy$$

$$= \int_0^1 \int_0^1 \left((u+2v) + (-u)\right)^2 \cdot |J(P)| \cdot dv \, du$$

$$= \int_0^1 \int_0^1 (2v)^2 (2) \, dv \, du$$

$$= \int_0^1 \int_0^1 8v^2 \, dv \, du$$

$$= \int_0^1 \frac{8}{3} v^3 \Big|_{v=0}^{v=1} du$$

$$= \int_0^1 \frac{8}{3} du = \frac{8}{3} u \Big|_0^1 = \frac{8}{3} \, .$$