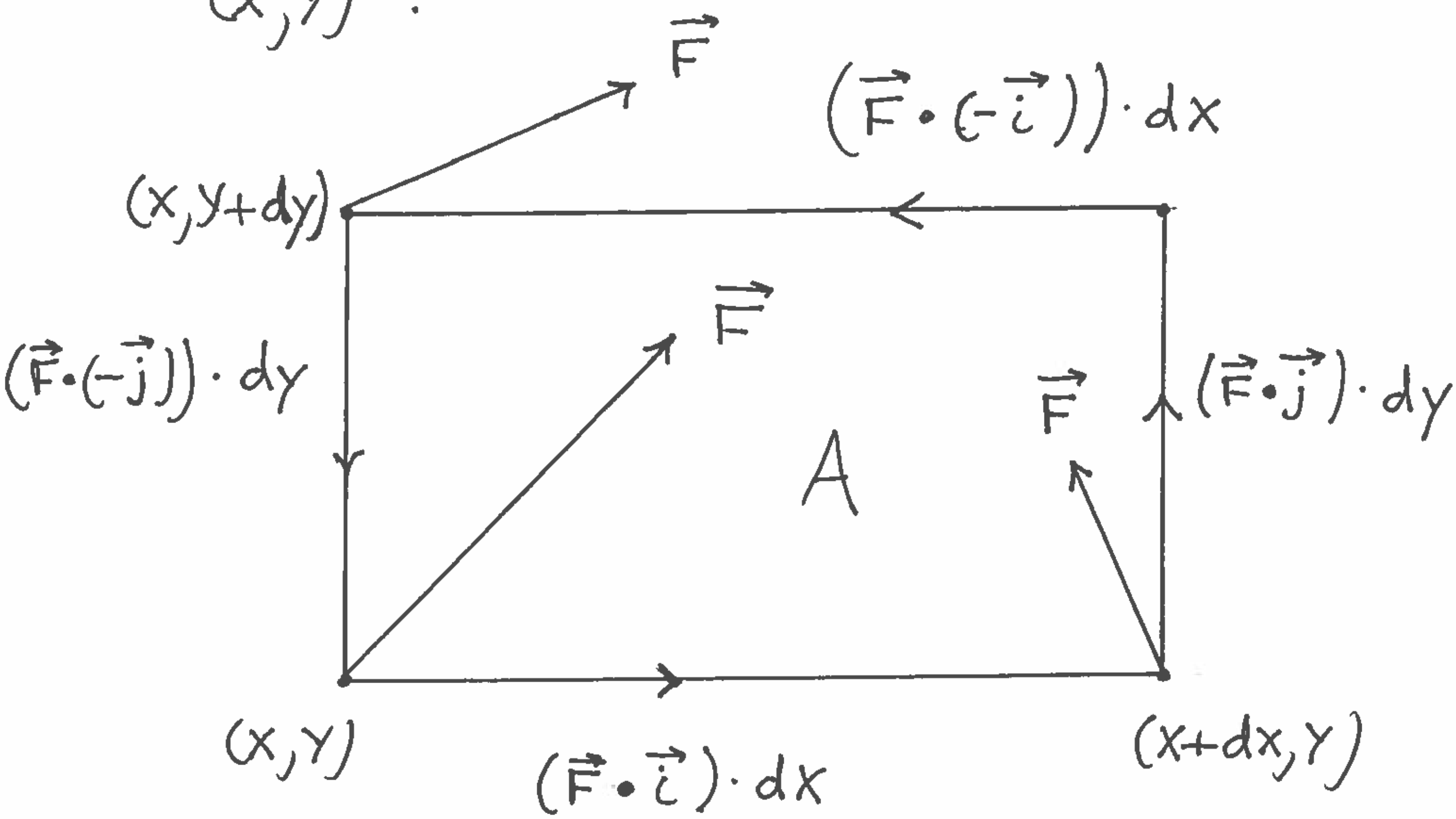
The k-Component of Curl, also called Circulation Density

These notes will help explain why the definition of the k-component of curl (circulation density) is a sensible one.

Assume that an incompressible fluid over a region R in the xy-plane is represented by a velocity vector field

$$\underbrace{\vec{F}(x,y) = \underset{\substack{\uparrow \\ \text{fluid density:} \\ \text{gm./cm.}^2}}{\delta} \cdot \underset{\substack{\uparrow \\ \text{fluid velocity:} \\ \text{cm./sec.}}}{\vec{V}(t)}}_{\frac{\text{gm.}}{(\text{cm.})(\text{sec.})}}$$

Consider a small rectangle A in region R at point (x,y) of dimensions dx by dy cm. Let's estimate the <u>circulation</u> of fluid at point (x,y) :



$$\underbrace{\frac{gm.}{(cm.)(sec.)}\cdot cm.}_{\frac{gm.}{sec.}}$$

assume $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$. We can now estimate the circulation rate of fluid around the boundary of A by adding the four rates of the four edges:

<u>edge</u>:

bottom: $(\vec{F} \cdot \vec{i}) \cdot dx = M(x,y) \cdot dx$

right: $(\vec{F} \cdot \vec{j})\, dy = N(x+dx,y) \cdot dy$

top: $(\vec{F} \cdot (-\vec{i}))\, dx = -M(x,y+dy) \cdot dx$

left: $(\vec{F} \cdot (-\vec{j}))\, dy = -N(x,y) \cdot dy$, then

circulation rate is $\approx$

$$N(x+dx,y) \cdot dy - N(x,y) \cdot dy$$

$$- \left(M(x,y+dy) \cdot dx - M(x,y) \cdot dx\right)$$

$$= \left(N(x+dx,y) - N(x,y)\right) \cdot dy$$

$$- \left(M(x,y+dy) - M(x,y)\right) \cdot dx$$

$$= \frac{N(x+dx,y) - M(x,y)}{dx} \cdot dx \cdot dy$$

$$- \frac{M(x,y+dy) - M(x,y)}{dy} \cdot dy \cdot dx$$

$$\approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy \text{ ; now divide}$$

by the area of A, getting

$$\text{circulation density} = \frac{\text{circulation around } A}{\text{area of } A}$$

$$= \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \cdot dx \cdot dy}{dx \cdot dy}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad \frac{\text{gm.}}{(\text{sec.})(\text{cm.}^2)}$$