Math 21D
Kouba

Gradient Fields (Conservative Vector Fields) and Path Independence

Recall: Consider the scalar function $w = f(x, y, z)$. Its gradient field is the vector field

$$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} .$$

Example A: If $f(x, y, z) = xy + yz + xz$, then its gradient field is

$$\vec{\nabla} f(x, y, z) = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k} .$$

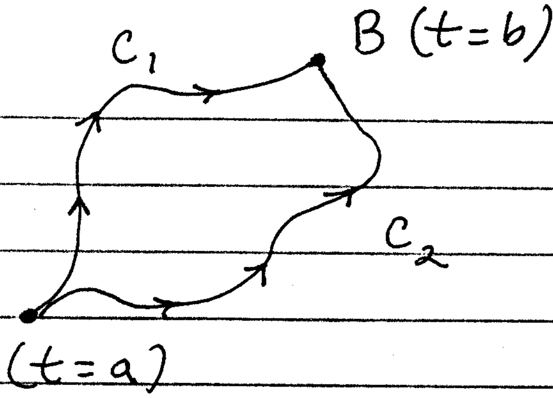
Of course not all vector fields are gradient fields. This is just one type of vector field. However, gradient fields have very special properties.

Recall: Let $\vec{F}$ be a vector field defined on path $C : \vec{r}(t)$ for $a \le t \le b$. The work done by $\vec{F}$ on path C is

$$\text{work} = \int_C \vec{F} \cdot \vec{T} \, ds .$$

Note: In general,

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds \neq \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$



for different paths C_1 and C_2 between points A and B. However, for some vector fields $\vec{F}$

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds.$$

for all paths C_1 and C_2 between points A and B. (See problem 12, p. 1142.)

Def: Let $\vec{F}$ be a vector field defined on a region D and let A and B be any two points in D. If

$$\int_{C_1} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$

for any two paths C_1 and C_2 from point A to point B, then we call $\vec{F}$ a conservative vector field. We say that the work integral from point A to point

B is <u>path independent</u>.

<u>Ex</u>: It will be shown later that the vector field in Example A is conservative.

<u>Recall</u>: (FTC from Math 21B)
If $f'(x) = F(x)$, then

$$\int_a^b F(x)\,dx = f(x)\Big|_a^b = f(b) - f(a).$$

<u>Recall</u>: (Chain Rule) If $w = f(x, y, z)$ and $x = g(t)$, $y = h(t)$, and $z = k(t)$, then

$$\frac{df}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} + f_z \cdot \frac{dz}{dt}.$$

<u>Fundamental Theorem for Line Integrals</u>: Let $w = f(x, y, z)$ be a scalar function and let $\vec{\nabla} f = f_x \cdot \vec{i} + f_y \cdot \vec{j} + f_z \cdot \vec{k}$ be its gradient field defined on path $C : \vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ for $a \leq t \leq b$. Let $A = \vec{r}(a)$ and $B = \vec{r}(b)$.
Then $\int_C \vec{\nabla} f \cdot \vec{T}\,ds = f(\vec{r}(t))\Big|_a^b = f(B) - f(A)$.

Proof: $\displaystyle\int_C \vec{\nabla} f(P)\cdot\vec{T}\,ds = \int_C \vec{\nabla} f(\vec{r}(t))\cdot\vec{r}(t)\,dt$

$\displaystyle = \int_a^b \left(f_x\vec{i}+f_y\vec{j}+f_z\vec{k}\right)\cdot\left(g'(t)\,\vec{i}+h'(t)\,\vec{j}+k'(t)\,\vec{k}\right)dt$

$\displaystyle = \int_a^b \left(f_x\cdot g'(t)+f_y\cdot h'(t)+f_z\cdot k'(t)\right)dt$

$\displaystyle = \int_a^b \frac{d}{dt} f(g(t),h(t),k(t))\,dt$ (Chain Rule)

$\displaystyle = f(g(t),h(t),k(t))\Big|_a^b$ (FTC)

$= f(g(b),h(b),k(b)) - f(g(a),h(a),k(a))$

$= f(B) - f(A)$.

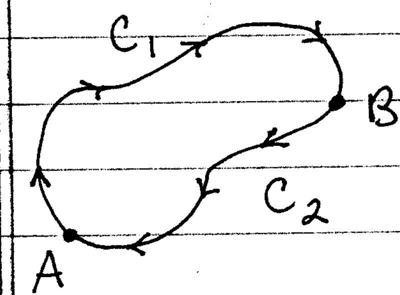
Fact: $\displaystyle\int_{t=a}^{t=b} f(P)\,ds = -\int_{t=b}^{t=a} f(P)\,ds$.

Theorem 1: A vector field $\vec{F}$ is conservative iff $\displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = 0$

for every closed curve C.

Proof: ($\Rightarrow$) Assume $\vec{F}$ is conservative. Show that $\displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = 0$ for every closed curve C. Let C be a closed path starting at point A. Let B be another point on path C. Let C_1 be the path from A to B and let C_2 be the path from B to A. Since $\vec{F}$ is conservative, we know that all work integrals from point A to point B are equal, so that

C



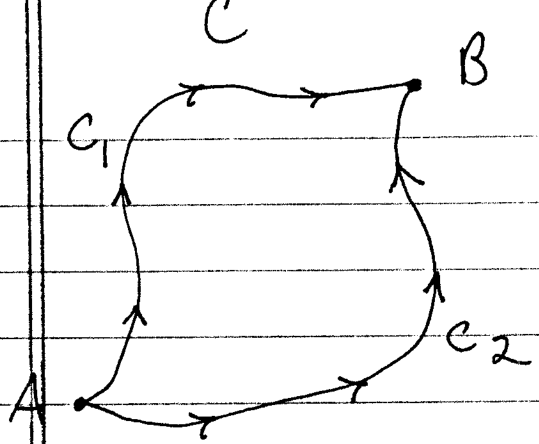
$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = -\int_{C_2} \vec{F}\cdot\vec{T}\,ds .$$

Then
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$
$$= -\int_{C_2} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds = 0 .$$

($\Leftarrow$) Assume $\oint_C \vec{F}\cdot\vec{T}\,ds = 0$ for every closed curve C. Let A and B be any two points. Show that $\vec{F}$ is conservative, i.e., show that

$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

for any two paths C_1 and C_2 from point A to point B. Consider the



closed path C from A to B (along C_1) and then from B to A (reverse of C_2). Then

$$0 = \oint_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds + \left(-\int_{C_2} \vec{F}\cdot\vec{T}\,ds\right)$$

$$\rightarrow \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds .$$

<u>Theorem 2</u>: Let $\vec{F}$ be a gradient field, i.e., assume that $\vec{F} = \vec{\nabla} f$ for some scalar function $w = f(x,y,z)$. Then $\vec{F}$ is conservative.

<u>Proof</u>: Let path C be given by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ for $a \le t \le b$, where $A = \vec{r}(a)$ and $B = \vec{r}(b)$. Then

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{\nabla} f\cdot\vec{T}\,ds$$

$$= f(\vec{r}(t))\Big|_a^b = f(B) - f(A).$$

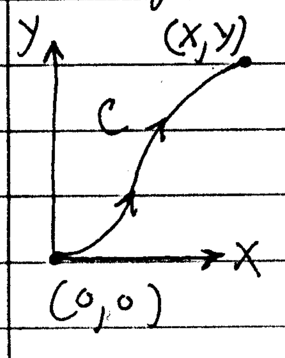
(by Fundamental Theorem for Line Integrals)

The answer implies that only the endpoints of path C matter, so that the work integral from point A to point B is path independent. This means $\vec{F}$ is conservative.

Theorem 3: Assume that $\vec{F}$ is a conservative vector field. Then $\vec{F}$ must also be a gradient field, i.e., there is some scalar function $w = f(x, y, z)$ so that

$$\vec{F} = \vec{\nabla} f .$$

Proof: (in 2D-space) Assume that $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is a conservative vector field. Define scalar function f as follows: for each point (x,y) select a path C from $(0,0)$ to (x,y). Define

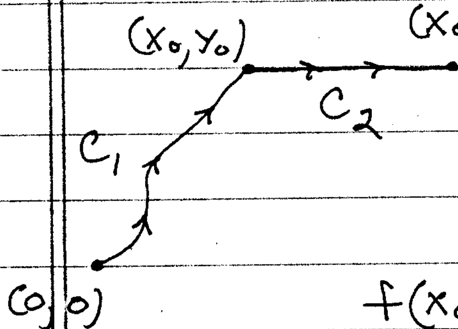


$$f(x,y) = \int_C \vec{F} \cdot \vec{T}\, ds .$$

We will show that $\vec{\nabla} f = \vec{F}$, i.e., that $f_x = M$ and $f_y = M$.

Let (x_0, y_0) be a fixed point and compute $\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$:

Let C_1 be any path from $(0,0)$ to (x_0, y_0) and let C_2 be the straight path from (x_0, y_0) to (x_0+h, x_0). Let C be paths C_1 and C_2 combined. Then



$$f(x_0, y_0) = \int_{C_1} \vec{F} \cdot \vec{T}\, ds \text{ , and}$$

$$f(x_0+h, y_0) = \int_{C} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} \vec{F} \cdot \vec{r}\,'(t)\, dt$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} (M\vec{i} + N\vec{j})\left(\frac{dx}{dt}\vec{i} + \cancelto{0}{\frac{dy}{dt}\vec{j}}\right) dt$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} M \frac{dx}{dt}\, dt$$

$$= \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{x_0}^{x_0+h} M(x, y_0)\, dx \text{ ; then}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{\int_{x_0}^{x_0+h} M(x, y_0)\, dx}{h} = M(x_0, y_0) \text{ ; i.e.,}$$

SEE (*) below

$\frac{\partial f}{\partial x}(x_0, y_0) = M(x_0, y_0)$; similarly,

$\frac{\partial f}{\partial y}(x_0, y_0) = N(x_0, y_0)$. This establishes that $\vec{F} = \vec{\nabla} f$.

(*) (Recall: (FTC from Math 21B)

$$\frac{d}{dx}\int_a^x G(t)\, dt = G(x) \text{ ; and}$$

$$\frac{d}{dx}\int_a^x G(t)\, dt = \lim_{h\to 0} \frac{\int_a^{x+h} G(t)\, dt - \int_a^x G(t)\, dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} G(t)\, dt}{h} = G(x).)$$

<u>Question</u>: Is there a test to determine if a given vector field $\vec{F}$ is conservative?

<u>Component Test for Conservative Fields</u>:

Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field. Then $\vec{F}$ is conservative iff

$$P_y = N_z, \quad M_z = P_x, \text{ and } N_x = M_y.$$

Proof: $(\Rightarrow)$ If $\vec{F}$ is conservative, then there is a scalar function f satisfying $\vec{F} = \vec{\nabla} f$; thus

$$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k} = f_x \cdot \vec{i} + f_y \cdot \vec{j} + f_z \cdot \vec{k}.$$

It follows that

$$P = f_z \rightarrow P_y = (f_z)_y = (f_y)_z = N_z;$$

$$M = f_x \rightarrow M_z = (f_x)_z = (f_z)_x = P_x;$$

$$N = f_y \rightarrow N_x = (f_y)_x = (f_x)_y = M_y.$$

$(\Leftarrow)$ Omitted.

Def: If $\vec{F}$ is a conservative vector field, then $\vec{F} = \vec{\nabla} f$ for some scalar function f. We call f a potential function for $\vec{F}$.

Ex: Show that each vector field $\vec{F}$ is conservative and then find a potential function f.

1.) $\vec{F}(x,y) = (xy^2)\vec{i} + (x^2y+1)\vec{j}$

2.) $\vec{F}(x,y,z) = (y)\vec{i} + (x)\vec{j} + (2z)\vec{k}$