Section 16.3
Thomas Calculus
11th Ed.

Gradient Fields (Conservative Vector Fields) and Path Independence

Recall: Consider the scalar function \( w = f(x, y, z) \). Its gradient field is the vector field
\[
\vec{\nabla}f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}
\]

Example I: \( f(x, y, z) = xy + yz + xz \), then its gradient field is
\[
\vec{\nabla}f(x, y, z) = (y + z)\hat{i} + (x + z)\hat{j} + (x + y)\hat{k}
\]

Of course, not all vector fields are gradient fields. This is just one type of vector field. However, gradient fields have very special properties.

Recall: Let \( \vec{F} \) be a vector field defined on path \( C : \vec{r}(t) \) for \( a \leq t \leq b \). The work done by \( \vec{F} \) on path \( C \) is
\[
\text{Work} = \int_C \vec{F} \cdot \vec{T} \, ds
\]
Note: In general,
\[ \int_{C_1} F \cdot T \, ds + \int_{C_2} F \cdot T \, ds = \int_{C_2} F \cdot T \, ds + \int_{C_1} F \cdot T \, ds \]
for different paths \( C_1 \) and \( C_2 \) between points \( A \) and \( B \). However, for some vector fields \( F \)
\[ \int_{C_1} F \cdot T \, ds = \int_{C_2} F \cdot T \, ds \]
for all paths \( C_1 \) and \( C_2 \) between points \( A \) and \( B \). (See problem 12, p. 1142.)

Def: Let \( F \) be a vector field defined on a region \( D \) and let \( A \) and \( B \)
be any two points in \( D \). If
\[ \int_{C_1} F \cdot T \, ds = \int_{C_2} F \cdot T \, ds \]
for any two paths \( C_1 \) and \( C_2 \)
from point \( A \) to point \( B \), then we call \( F \) a conservative vector field. We say that the work integral from point \( A \) to point
B is path independent.

**Ex:** It will be shown later that the vector field in Example A is conservative.

Recall: (FTC from Math 21B)

If \( f'(x) = F(x) \), then

\[
\int_a^b F(x) \, dx = f(x) \bigg|_a^b = f(b) - f(a).
\]

Recall: (Chain Rule) If \( w = f(x, y, z) \) and \( x = g(t) \), \( y = h(t) \), and \( z = k(t) \), then

\[
\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}.
\]

Fundamental Theorem for Line Integrals: Let \( w = f(x, y, z) \) be a scalar function and let \( \mathbf{F} = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k} \) be its gradient field defined on path \( \mathbf{C} \) with \( \mathbf{C} : \mathbf{r}(t) = g(t) \mathbf{i} + h(t) \mathbf{j} + k(t) \mathbf{k} \) for \( a \leq t \leq b \). Let \( A = \mathbf{r}(a) \) and \( B = \mathbf{r}(b) \). Then

\[
\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{r}(b)) \bigg|_a^b = f(B) - f(A).
\]
\[ \text{Proof: } \int_C \nabla f(p) \cdot \vec{T} \, ds = \int_C \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]
\[ = \int_a^b (f_x \vec{i} + f_y \vec{j} + f_z \vec{k}) \cdot (g'(t) \vec{i} + h'(t) \vec{j} + k'(t) \vec{k}) \, dt \]
\[ = \int_a^b (f_x \cdot g'(t) + f_y \cdot h'(t) + f_z \cdot k'(t)) \, dt \]
\[ = \int_a^b \frac{d}{dt} f(g(t), h(t), k(t)) \, dt \quad \text{ (Chain Rule)} \]
\[ = f(g(b), h(b), k(b)) - f(g(a), h(a), k(a)) \]
\[ = f(B) - f(A) \]

Fact: \[ \int_{t=a}^{t=b} f(p) \, ds = -\int_{t=b}^{t=a} f(p) \, ds \]

Theorem 1: A vector field \( \vec{F} \) is conservative iff \[ \int_C \vec{F} \cdot \vec{T} \, ds = 0 \]
for every closed curve \( C \).

Proof: (\( \Rightarrow \)) Assume \( \vec{F} \) is conservative. Show that \[ \int_C \vec{F} \cdot \vec{T} \, ds = 0 \] for every closed curve \( C \). Let \( C \) be a closed...
path starting at point A. Let B be another point on path C. Let \( C_1 \) be the path from A to B and let \( C_2 \) be the path from B to A. Since \( \vec{F} \) is conservative, we know that all work integrals from point A to point B are equal, so that
\[
\int_{C_1} \vec{F} \cdot d\vec{s} = -\int_{C_2} \vec{F} \cdot d\vec{s}.
\]
Then
\[
\int_{C} \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} = -\int_{C_2} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} = 0.
\]
(\( \Leftarrow \) assume \( \int_{C} \vec{F} \cdot d\vec{s} = 0 \) for every closed curve \( C \). Let A and B be any two points. Show that \( \vec{F} \) is conservative, i.e., show that
\[
\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}
\]
for any two paths \( C_1 \) and \( C_2 \) from point A to point B. Consider the
closed path $C$
from $A$ to $B$ (along $C_1$) and then from $B$ to $A$ (reverse of $C_2$). Then
\[ 0 = \oint_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \left( -\int_{C_2} \mathbf{F} \cdot d\mathbf{s} \right) \]
\[ \rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}. \]

**Theorem 2**: Let $\mathbf{F}$ be a gradient field, i.e., assume that $\mathbf{F} = \nabla f$ for some scalar function $w = f(x, y, z)$. Then $\mathbf{F}$ is conservative.

**Proof**: Let path $C$ be given by
$\mathbf{r}(t) = g(t) \mathbf{i} + h(t) \mathbf{j} + k(t) \mathbf{k}$ for $a \leq t \leq b$,
where $\mathbf{A} = \mathbf{r}(a)$ and $\mathbf{B} = \mathbf{r}(b)$. Then
\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \nabla f \cdot d\mathbf{s} \]
\[ = f(\mathbf{r}(t)) \bigg|_a^b = f(\mathbf{B}) - f(\mathbf{A}). \]
(by Fundamental Theorem for Line Integrals)
The answer implies that only the endpoints of path $C$ matter, so that the work integral from point $A$ to point $B$ is path independent. This means $\mathbf{F}$ is conservative.

**Theorem 3:** Assume that $\mathbf{F}$ is a conservative vector field. Then $\mathbf{F}$ must also be a gradient field, i.e., there is some scalar function $\varphi = \varphi(x, y, z)$ so that

$$\mathbf{F} = \nabla \varphi.$$

**Proof:** (in 2D-space) Assume that $\mathbf{F}(x, y) = M(x, y) \hat{i} + N(x, y) \hat{j}$ is a conservative vector field. Define scalar function $\varphi$ as follows: for each point $(x, y)$ select a path $C$ from $(0, 0)$ to $(x, y)$. Define

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{s}.$$

We will show that $\nabla \varphi = \mathbf{F}$ at $(0, 0)$, i.e., that $f_x = M$ and $f_y = N$. 

7
Let \((x_0, y_0)\) be a fixed point and compute \(\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}\).

Let \(C_1\) be any path from \((0, 0)\) to \((x_0, y_0)\) and let \(C_2\) be the straight path from \((x_0, y_0)\) to \((x_0 + h, y_0)\). Let \(C\) be the combined paths \(C_1\) and \(C_2\) combined. Then

\[
\begin{align*}
    f(x_0, y_0) & = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds, \\
    f(x_0 + h, y_0) & = \int_{C} \mathbf{F} \cdot \mathbf{T} \, ds \\
    & = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{T} \, ds \\
    & = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n}'(t) \, dt \\
    & = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds + \int_{C_2} (M \hat{c} + N \hat{j}) \left( \frac{dx}{dt} \hat{c} + \frac{dy}{dt} \hat{j} \right) \, dt \\
    & = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds + \int_{C_2} M \frac{dx}{dt} \, dt \\
    & = \int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds + \int_{x_0}^{x_0 + h} M(x, y_0) \, dx \
\end{align*}
\]

Thus, \(\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}\).
\[ \lim_{h \to 0} \frac{\int_{x_0}^{x_0+h} M(x, y) \, dx}{h} = M(x_0, y_0) \text{ i.e.,} \]

\[ \frac{\partial f}{\partial x}(x_0, y_0) = M(x_0, y_0); \text{ similarly,} \]

\[ \frac{\partial f}{\partial y}(x_0, y_0) = N(x_0, y_0). \text{ This establishes that } \vec{F} = \nabla f. \]

\[ (\ast) \text{ (Recall: FTC from Math 21B)} \]

\[ \frac{d}{dx} \int_{a}^{x} G(t) \, dt = G(x) \text{ and} \]

\[ \frac{d}{dx} \int_{a}^{x} G(t) \, dt = \lim_{h \to 0} \frac{\int_{a}^{x+h} G(t) \, dt - \int_{a}^{x} G(t) \, dt}{h} \]

\[ = \lim_{h \to 0} \frac{\int_{x}^{x+h} G(t) \, dt}{h} = G(x). \]

**Question:** Is there a test to determine if a given vector field \( \vec{F} \) is conservative? ... YES!

**Component Test for Conservative Fields:**

1) Let

\[ \vec{F}(x, y, z) = M(x, y, z) \vec{i} + N(x, y, z) \vec{j} + P(x, y, z) \vec{k} \]
be a vector field. Then \( \vec{F} \) is CONSERVATIVE iff

\[ P_y = N_z, \quad M_z = P_x, \quad \text{and} \quad N_x = M_y. \]

II. Let \( \vec{F}(x,y) = M(x,y) \hat{i} + N(x,y) \hat{j} \) be a vector field. Then \( \vec{F} \) is CONSERVATIVE iff

\[ N_x = M_y. \]

Proof: (\( \Rightarrow \)) If \( \vec{F} \) is conservative, then there is a scalar function \( f \) satisfying \( \vec{F} = \nabla f \); thus

\[ \vec{F} = M \hat{i} + N \hat{j} + P \hat{k} = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}. \]

It follows that

\[ P = f_z \rightarrow P_y = (f_z)_y = (f_y)_z = N_z; \]

\[ M = f_x \rightarrow M_z = (f_x)_z = (f_z)_x = P_x; \]

\[ N = f_y \rightarrow N_x = (f_y)_x = (f_x)_y = M_y. \]

(\( \Leftarrow \)) Omitted.

II. If \( \vec{F}(x,y) = M(x,y) \hat{i} + N(x,y) \hat{j} \)
then \( N_z = 0 \), \( M_z = 0 \), and \( P = 0 \) so that \( \rho_x = \rho_y = 0 \). The result follows.

**Definition**: We know that if \( \vec{F} \) is a conservative vector field, then \( \vec{F} = \nabla f \) for some scalar function \( f \). We call \( f \) a potential function for \( \vec{F} \).

**Example**: Show that each vector field \( \vec{F} \) is conservative and then find a potential function \( f \).

1. \( \vec{F}(x,y) = (xy^2) \vec{i} + (x^2y + 1) \vec{j} \)
2. \( \vec{F}(x,y,z) = (y) \vec{i} + (x) \vec{j} + (az) \vec{k} \)
3. \( \vec{F}(x, y) = (e^x \sin y + \tan y) \hat{i} + (e^x \cos y + x \sec^2 y) \hat{j} \)

4. \( \vec{F}(x, y, z) = \left( \frac{2x + 3x^2y^2}{z^2} \right) \hat{i} \\
+ \left( \frac{2x^3y}{z^2} \right) \hat{j} - \left( \frac{1 + 2x^2 + 2x^3y^2}{z^3} \right) \hat{k} \)

1. \( \vec{F}(x, y) = (xy^2) \hat{i} + (x^2y + 1) \hat{j} \), then

\[ M_y = 2xy = N_x \], so this vector field is conservative; so

\[ f_x = xy^2 \rightarrow f = \frac{1}{2} x^2 y^2 + g(y) \]

\[ f_y = x^2 y + g'(y) = x^2 y + 1 \rightarrow \]

\[ g'(y) = 1 \rightarrow g(y) = y + C_0 \], so

\[ f(x, y) = \frac{1}{2} x^2 y^2 + y \] is a potential function.
3.) \( \vec{F}(x,y) = (e^x \sin y + \tan y) \hat{i} + (e^x \cos y + x \sec^2 y) \hat{j} \),

then

\[ M_y = e^x \cos y + \sec^2 y = N_x, \]

so this vector field is conservative; so

\[ f_x = e^x \sin y + \tan y \quad \Rightarrow \quad \nabla \times \vec{F} \]

\[ f = e^x \sin y + x \tan y + g(y) \quad \Rightarrow \quad \nabla \cdot \vec{F} \]

\[ f_y = e^x \cos y + x \sec^2 y + g'(y) \]

\[ = e^x \cos y + x \sec^2 y \rightarrow g'(y) = 0 \rightarrow \]

\[ g(y) = e^0, \text{ then } \]

\[ f(x,y) = e^x \sin y + x \tan y \]

is a potential function.

2.) \( \vec{F}(x,y,z) = (y) \hat{i} + (x+1) \hat{j} + (2z-3) \hat{k} \),

then

\[ P_y = 0 = N_z, \quad M_z = 0 = P_x, \text{ and } N_x = 1 = M_y, \]

so this vector field is conservative;
\[ P_y = -\frac{0 + 2x^3(2y)}{z^3} = -\frac{4x^3y}{z^3} \quad \text{and} \]
\[ N_z = 2x^3y \cdot (-2z^{-3}) = -\frac{4x^3y}{z^3}, \text{so} \]
\[ P_y = N_z \quad j \]
\[ N_x = \frac{6x^2y}{z^2} \quad \text{and} \]
\[ M_y = \frac{0 + 3x^2(2y)}{z^2} = \frac{6x^2y}{z^2}, \text{so} \]
\[ N_x = M_y \quad j \]
\[ M_z = (2x + 3x^2y^2) \cdot (-2z^{-3}) \]
\[ = -\frac{4x - 6x^2y^2}{z^3} \quad \text{and} \]
\[ P_x = -\frac{4x + 6x^2y^2}{z^3}, \text{so} \]
\[ M_z = P_x \quad j \quad \text{so this vector field is conservative.} \]
\[ f_x = y \quad \overset{S_x}{\Rightarrow} \quad f = xy + g(y, z) \quad \overset{D_y}{\Rightarrow} \quad s_y \]

\[ f_y = x + g_y(y, z) = x + 1 \quad \rightarrow \quad g_y(y, z) = 1 \quad \overset{D_2}{\Rightarrow} \quad f_z = 0 + 0 + k'(z) = 2z - 3 \quad \overset{S}{\Rightarrow} \quad k(z) = z^2 - 3z + e^0, \]

then

\[ f(x, y, z) = xy + y + z^2 - 3z \]

is a potential function.

4.) \[ \vec{F}(x, y, z) = \left( \frac{2x + 3x^2y^2}{z^2} \right) \hat{i} \]

\[ + \left( \frac{2x^3y}{z^2} \right) \hat{j} - \left( \frac{1 + 2x^2 + 2x^3y^2}{z^3} \right) \hat{k} \]

with

\[ M = \frac{2x + 3x^2y^2}{z^2}, \quad N = \frac{2x^3y}{z^2}, \quad \text{and} \]

\[ P = - \frac{2x^2 + 2x^3y^2}{z^3} \]

then
\[ f_y = \frac{2x^3y}{z^2} \quad S_y \rightarrow f = \frac{2x^3 \cdot \frac{1}{2} y^2}{z^2} + g(x, z) \]

\[ = \frac{x^3 y^2}{z^2} + g(x, z) \rightarrow D_x \]

\[ f_x = \frac{3x^2y^2}{z^2} + g_x(x, z) \]

\[ = \frac{2x}{z^2} + \frac{3x^2 y^2}{z^2} \rightarrow g_x(x, z) = \frac{2x}{z^2} \]

\[ S_x \rightarrow g(x, z) = \frac{x^2}{z^2} + k(z) \rightarrow \]

\[ f = \frac{x^3 y^2}{z^2} + \frac{x^2}{z^2} + k(z) \rightarrow D_z \]

\[ f_z = x^3 y^2 (-2z^{-3}) + x^2 (-2z^{-3}) + k'(z) \]

\[ = \frac{-2x^3 y^2}{z^3} + \frac{-2x^2}{z^3} + k'(z) \]

\[ = -\frac{1}{z^3} - \frac{2x^2}{z^3} + \frac{-2x^3 y^2}{z^3} \rightarrow \]

\[ k'(z) = \frac{-1}{z^3} = -z^{-3} \rightarrow k(z) = \frac{1}{2} z^{-2} + \phi_0 \]
\[ f(x, y, z) = \frac{x^3 y^2 + x^2}{z^2} + \frac{1}{2z^2} \]

is a potential function.

**Example:** Consider the vector field

\[ \mathbf{F}(x, y, z) = (y + z)i + (x + z)j + (x + y)k \]

and the path \( C \) in 3D space described as follows:

\( C \) starts at point \((1, 0, 0)\), follows the graph of \( y = \ln x \) in the \( xy \)-plane to point \((e^2, 2, 0)\), and then follows a straight line to the point \((0, 3, 4)\), where path \( C \) ends. Compute the work done by \( \mathbf{F} \) along path \( C \).
Here is path C:

Let's first check to see if $\vec{F}$ is a conservative vector field:

$$\vec{F}(x, y, z) = (y + z) \hat{i} + (x + z) \hat{j} + (x + y) \hat{k}$$

$$p_y = 1 = N_z, \quad N_x = 1 = M_y, \quad \text{and} \quad M_z = 1 = P_x,$$

so $\vec{F}$ is conservative. Now let's find a potential function $f$ and apply The Fundamental
Theorem for line integrals.

Then

\[ \mathbf{f}_x = y + z \Rightarrow f = xy + xz + g(y, z) \]

\[ \mathbf{D}_x f_y = x + o + g_y(y, z) = x + z \Rightarrow \]

\[ g_y(y, z) = z \Rightarrow g(y, z) = yz + k(z) \]

\[ \Rightarrow f = xy + xz + yz + k(z) \Rightarrow \]

\[ \mathbf{f}_z = o + x + y + k'(z) = x + y \Rightarrow \]

\[ k'(z) = 0 \Rightarrow k(z) = c, \text{ then} \]

\[ f(x, y, z) = xy + xz + yz \text{; apply Theorem getting} \]

Work = \[ \int_C \mathbf{F} \cdot \mathbf{T} \, ds \]

\[ = f(x, y, z) \bigg|^{(0, 3, 4)}_{(1, 0, 0)} \]

\[ = [(0)(3) + (0)(4) + (3)(4)] - [(1)(0) + (1)(0) + (0)(0)] = 12 \]