Section 13.4
Thomas Calculus
11th Ed.

Curvature of Path C in 2D-Space or 3D-Space

Consider a path C determined by vector function $\vec{r}(t)$, its velocity $\vec{v}(t)$, and arc length s.

<u>Definition</u>: Let $\vec{r}(t)$ be a vector function and $\vec{T}(t)$ its unit tangent vector. The <u>curvature</u> of path C at a point P is

$$K = \left| \frac{d\vec{T}}{ds} \right| .$$

More Practical Formula:

We can rewrite the curvature

formula using different derivative forms.

$$K = \left|\frac{d\vec{T}}{ds}\right|$$

$$= \left|\frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}\right|$$

$$= \left|\frac{d\vec{T}}{dt}\cdot\frac{1}{\frac{ds}{dt}}\right|$$

$$= \frac{1}{\left|\frac{ds}{dt}\right|}\cdot\left|\vec{T}'(t)\right|$$, i.e.,

$$\boxed{K = \frac{1}{|\vec{v}(t)|}\cdot\left|\vec{T}'(t)\right|}\ .$$

<u>Example</u> : Consider path C determined by the vector function

$$\vec{r}(t) = (t)\vec{i} + \left(\tfrac{1}{2}t^2\right)\vec{j}\ .$$

1.) Plot path C.

2.) Find the curvature formula for path C.

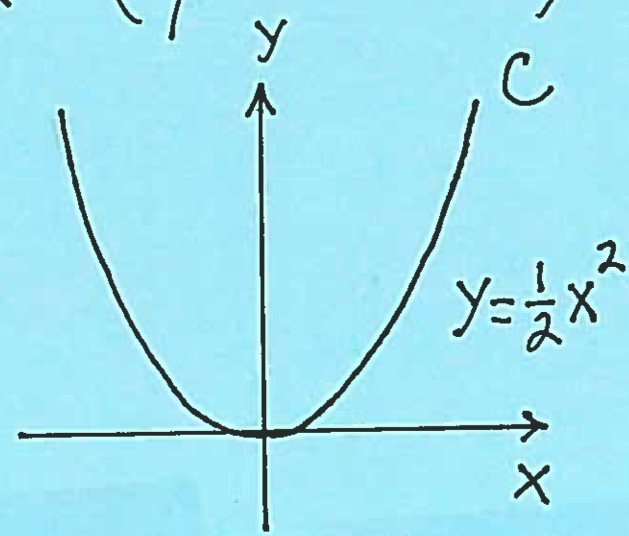
a.) Find K for $t=0$.

b.) Find K for $t=1$.

c.) Find K for $t=10$.

1.) $\vec{r} = (t)\vec{i} + (\frac{1}{2}t^2)\vec{j} \longrightarrow$

$$\begin{cases} x = t \\ y = \frac{1}{2}t^2 \end{cases} \longrightarrow y = \frac{1}{2}x^2 \text{ (parabola)}$$



2.) $\vec{v}(t) = (1)\vec{i} + (t)\vec{j} \longrightarrow$

$|\vec{v}(t)| = \sqrt{1^2 + t^2} \longrightarrow$

$\boxed{|\vec{v}(t)| = \sqrt{1+t^2}}$; so

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{1+t^2}}\vec{i} + \frac{t}{\sqrt{1+t^2}}\vec{j}$$

$$= (1+t^2)^{-1/2}\,\vec{i} + \frac{t}{\sqrt{1+t^2}}\,\vec{j} \xrightarrow{D}$$

$$\vec{T}'(t) = -\frac{1}{\cancel{2}}(1+t^2)^{-3/2}\cdot(\cancel{2}t)\cdot\vec{i}$$

$$+ \frac{\sqrt{1+t^2}\,(1) - t\cdot\frac{1}{\cancel{2}}(1+t^2)^{-1/2}\cdot(\cancel{2}t)}{\left(\sqrt{1+t^2}\right)^2}\,\vec{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\,\vec{i} + \frac{\frac{\sqrt{1+t^2}}{1} - \frac{t^2}{\sqrt{1+t^2}}}{1+t^2}\,\vec{j}$$

$$= \frac{-t}{(1+t^2)^{3/2}}\,\vec{i} + \frac{(1+t^2)-t^2}{\sqrt{1+t^2}}\cdot\frac{1}{1+t^2}\,\vec{j},$$ i.e.,

$$\boxed{\vec{T}'(t) = \frac{-t}{(1+t^2)^{3/2}}\,\vec{i} + \frac{1}{(1+t^2)^{3/2}}\,\vec{j}}$$; then

$$|\vec{T}'(t)| = \sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+t^2)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}}$$

$$= \sqrt{\frac{1+t^2}{(1+t^2)^3}} = \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2} \text{ ; i.e.,}$$

$$\boxed{|\vec{T}\,'(t)| = \frac{1}{1+t^2}} \quad .$$ Then

$$K = \frac{|\vec{T}\,'(t)|}{|\vec{v}(t)|} = \frac{\frac{1}{1+t^2}}{(1+t^2)^{1/2}}$$

$$= \frac{1}{1+t^2} \cdot \frac{1}{(1+t^2)^{1/2}} = \frac{1}{(1+t^2)^{3/2}} \text{ , i.e.,}$$

the curvature K at time t is

$$\boxed{K = \frac{1}{(1+t^2)^{3/2}}} \quad .$$

a.) If $t=0$, then curvature

$$K = \frac{1}{(1+0^2)^{3/2}} = 1 \quad .$$

b.) If $t=1$, then curvature

$$K = \frac{1}{(1+1^2)^{3/2}} = \frac{1}{2^{3/2}} \approx 0.3535$$

c.) If $t=10$, then curvature

$$K = \frac{1}{(1+10^2)^{3/2}} = \frac{1}{101^{3/2}} \approx 0.00098$$

<u>Example</u>: Consider path C determined by the vector function

$$\vec{r}(t) = (2\cos 3t)\vec{i} + (2\sin 3t)\vec{j} .$$

1.) Plot path C.

2.) Find the curvature formula for path C.

a.) Find K for $t=0$.

b.) Find K for $t=\frac{\pi}{2}$.

1.) $\vec{r}(t) = (2\cos 3t)\vec{i} + (2\sin 3t)\vec{j} \longrightarrow$

$$\begin{cases} x = 2\cos 3t \\ y = 2\sin 3t \end{cases} \longrightarrow$$

$$x^2+y^2 = (2\cos 3t)^2 + (2\sin 3t)^2$$

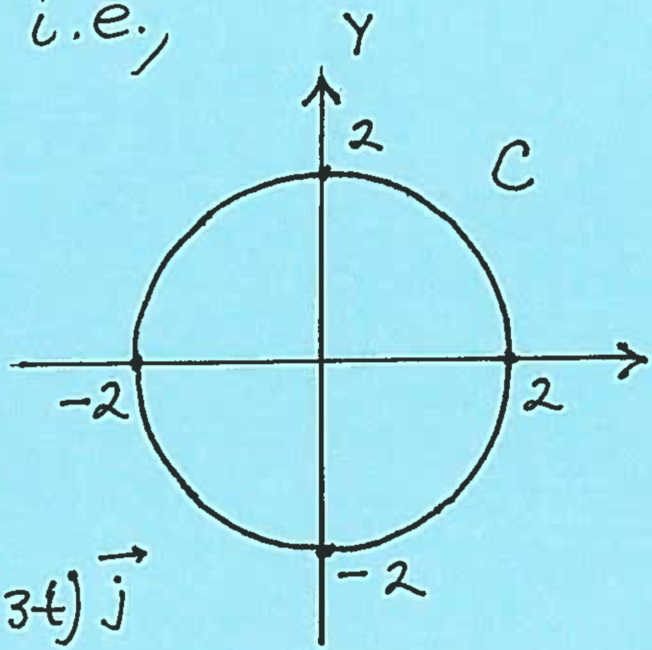
$= 4\cos^2 3t + 4\sin^2 3t$

$= 4(\cos^2 3t + \sin^2 3t)$

$= 4(1) = 4$, i.e.,

$x^2 + y^2 = 2^2$

(circle)



2.)

$\vec{v}(t) = (-6\sin 3t)\vec{i} + (6\cos 3t)\vec{j}$

$\rightarrow |\vec{v}(t)| = \sqrt{(-6\sin 3t)^2 + (6\cos 3t)^2}$

$= \sqrt{36\sin^2 3t + 36\cos^2 3t}$

$= \sqrt{36(\sin^2 3t + \cos^2 3t)} = 6(1) = 6$, i.e.,

$\boxed{|\vec{v}(t)| = 6}$; so

$\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = (-\sin 3t)\vec{i} + (\cos 3t)\vec{j} \xrightarrow{D}$

$\vec{T}'(t) = (-3\cos 3t)\vec{i} + (-3\sin 3t)\vec{j} \rightarrow$

$$|\vec{T}'(t)| = \sqrt{(-3\cos 3t)^2 + (-3\sin 3t)^2}$$

$$= \sqrt{9\cos^2 3t + 9\sin^2 3t}$$

$$= \sqrt{9(\cos^2 3t + \sin^2 3t)}$$

$$= \sqrt{9\cdot(1)} = 3 \text{, i.e.,}$$

$$\boxed{|\vec{T}'(t)| = 3} \text{ ; then}$$

$$K = \frac{|\vec{T}'(t)|}{|\vec{v}(t)|} = \frac{3}{6} = \frac{1}{2} \text{, i.e.,}$$

the curvature K at time t is

$$\boxed{K = \frac{1}{2}}$$

a.) If $t=0$, then curvature

$K = 1/2$.

b.) If $t = \frac{\pi}{2}$, then curvature

$K = 1/2$.

Fact: Every circle of radius a, has curvature

$$K = \frac{1}{a}$$

Circles of Curvature
(also called Osculating Circles)

Consider a path C determined by a vector function $\vec{r}(t)$.

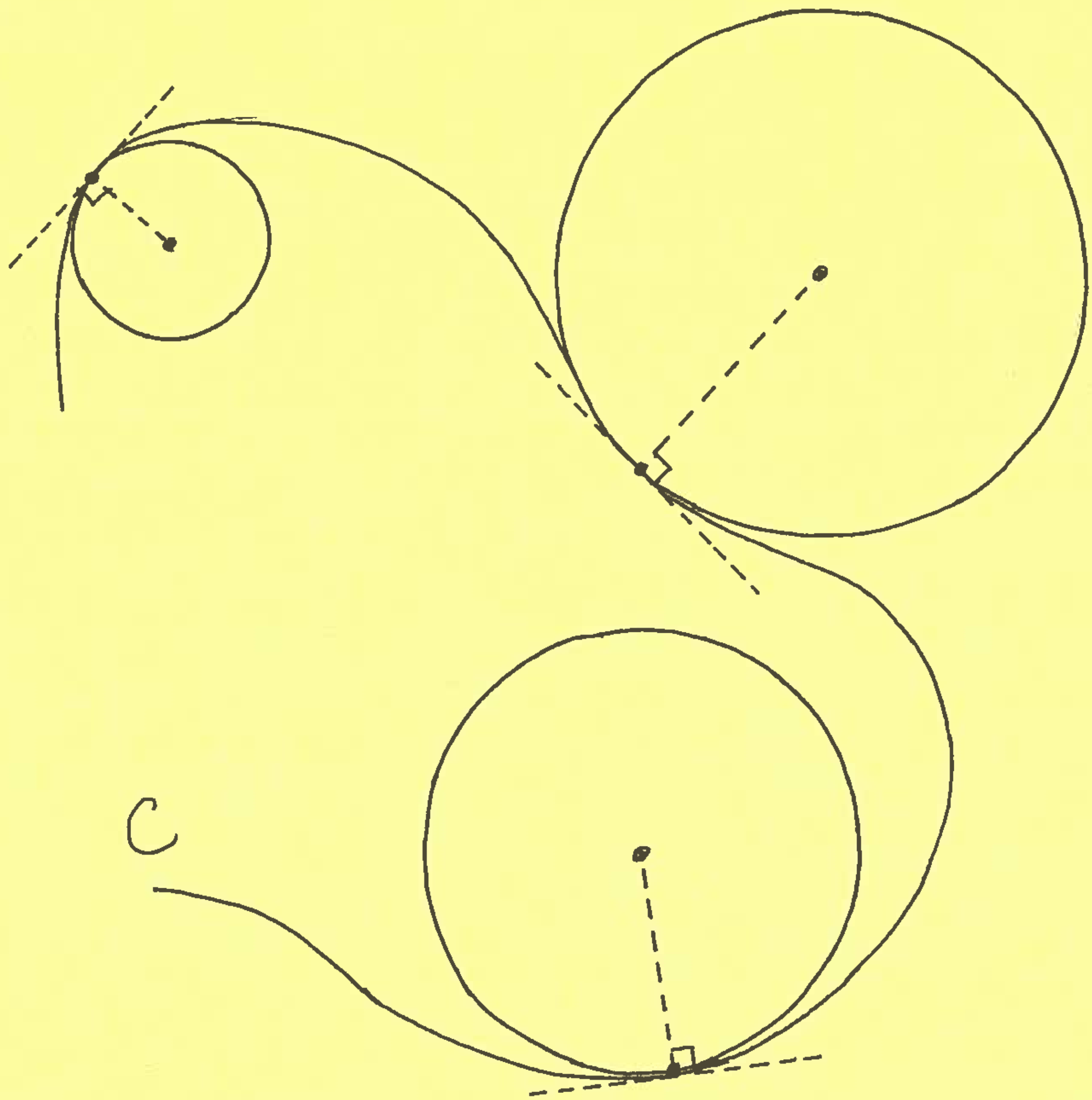
Definition: A Circle of Curvature to path C at point P is that circle which is

1.) tangent to path C at point P,

2.) on the concave side of path P,

and 3.) has the same curvature K as path C at point P.

Here is what Circles of Curvature look like.



Note: The "sharper" the turn, the smaller the Circle of Curvature.

The following problem illustrates the importance of curvature and the radius of curvature (the radius of the circle of curvature).

Choose a problem from chapter 6 31PE

Change the chapter (/choose-a-c

Question

Modern roller coasters have vertical loops like the one shown in Figure 6.38. The radius of cur smaller at the top than on the sides so that the downward centripetal acceleration at the top w greater than the acceleration due to gravity, keeping the passengers pressed firmly into their s What is the speed of the roller coaster at the top of the loop if the radius of curvature there is and the downward acceleration of the car is 1.50 g?

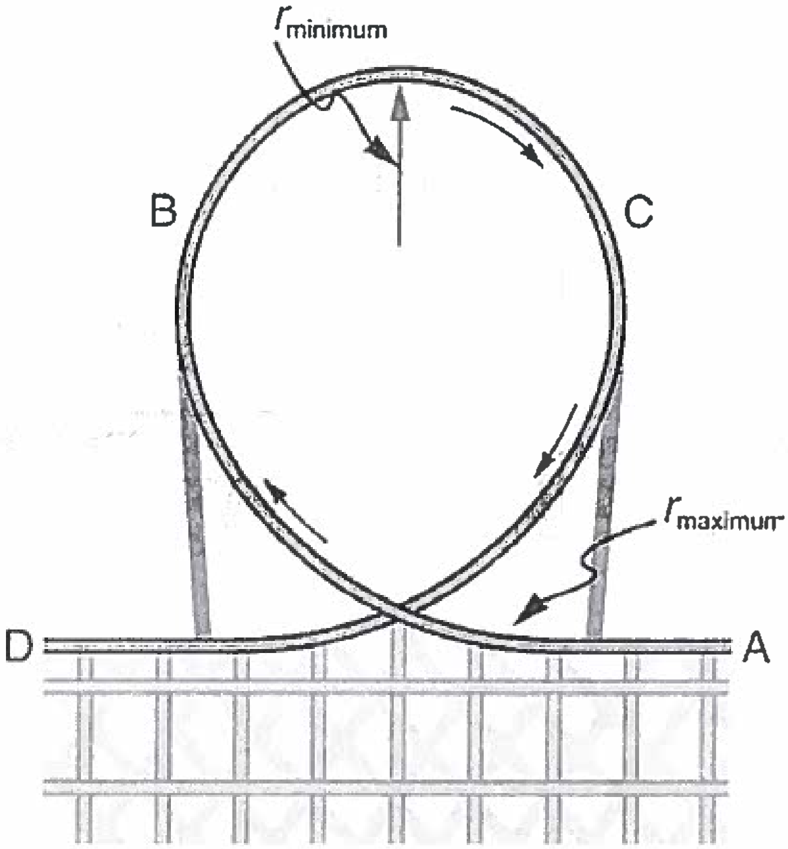


Figure 6.38 Teardrop-shaped loops are used in the latest roller coasters so that the radius of curvature gradu decreases to a minimum at the top. This means that the centripetal acceleration builds from zero to a maxim top and gradually decreases again. A circular loop would cause a jolting change in acceleration at entry, a dis discovered long ago in railroad curve design.