Where does the formula for Moment of Inertia come from?

Consider the top view of a flat plate in region $R$ rotating about an axis at an angular speed of $\omega$ radians/sec.

Assume that the density at point $P$ is $\delta(P)$ grams/cm$^2$, and
consider a small piece of the plate of mass $dm$ grams at a distance of $r$ cm. from the axis of rotation. Its velocity is

$$\left( \frac{\omega}{\text{rad./sec.}} \right) \left( \frac{2\pi r \text{ cm.}}{2\pi \text{ rad.}} \right) = \omega r \text{ cm./sec.}$$

Now the Kinetic Energy of the small piece is

$$K.E. = \frac{1}{2} (\text{mass})(\text{velocity})^2$$

$$= \frac{1}{2} (dm)(\omega r)^2$$

$$= \frac{1}{2} \omega^2 \cdot r^2 \, dm$$

$$= \frac{1}{2} \omega^2 \cdot r^2 \cdot \delta(p) \, dA,$$

so that the Total Kinetic Energy
of the plate is

$$K.E. = \int \int \frac{1}{2} \omega^2 r^2 \delta(p) \, dA$$

\[= \frac{1}{2} \omega^2 \int \int r^2 \delta(p) \, dA.\]

We define the Moment of Inertia to be:

$$M.I. = \int \int r^2 \delta(p) \, dA$$