Math 21D
Koubа
Divergence Theorem

<u>Recall</u>: (2D – Space) If
$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is a vector field defined on region R, then the divergence of $\vec{F}$ at (x,y) is the scalar function

$$div\ \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

<u>Def</u>: Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field defined on solid D. The <u>divergence of $\vec{F}$</u> at (x,y,z) is the scalar function

$$div\ \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

<u>Remark</u>: It can be shown that the divergence of $\vec{F}$ at point (x,y,z) is a measure of the tendency of moving fluid to accumulate or disperse at point (x,y,z).

Ex: Find the divergence for

$$\vec{F}(x,y,z) = (\underbrace{e^{x-z}\cos y}_{M})\vec{i} + (\underbrace{-e^{x-z}\sin y}_{N})\vec{j} + (\underbrace{x^2 y \tan z}_{P})\vec{k}:$$

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

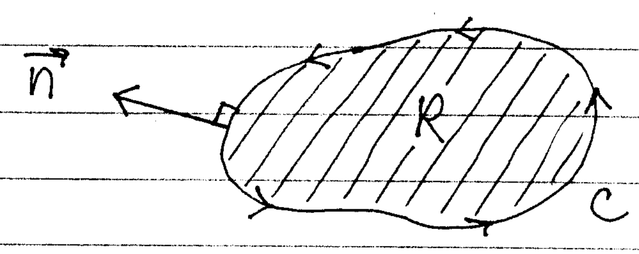
$$= \cancel{e^{x-z}\cos y} - \cancel{e^{x-z}\cos y} + x^2 y \sec^2 z$$

$$= x^2 y \sec^2 z .$$

Flux and Divergence in 3D-Space

Recall: (Green's Theorem – Flux Divergence Form – in 2D-Space) Assume that $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is defined on region R enclosed by loop C (oriented counter-clockwise) in 2D-Space. Then the flux of $\vec{F}$ across C is

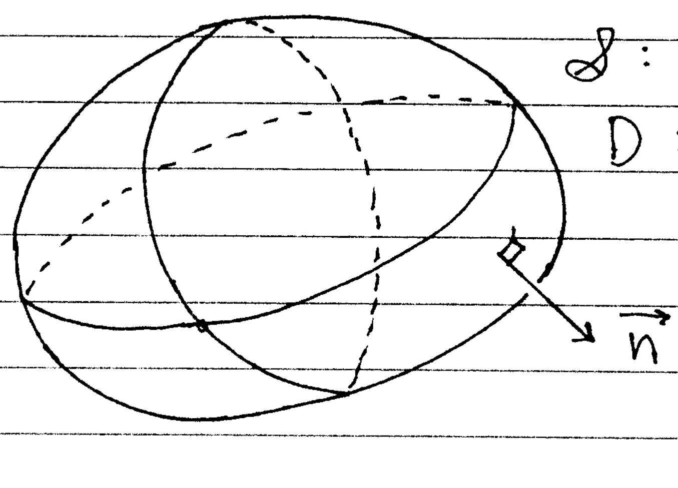
$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA .$$



The Divergence Theorem will generalize this concept in 3D-Space.

__Divergence Theorem__ : Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field defined on a solid D, enclosed by surface S with outward-pointing unit normal vector $\vec{n}$. Then the flux of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \underbrace{\vec{\nabla}\cdot\vec{F}}_{\operatorname{div}\vec{F}}\,dV .$$



S : surface
D : solid

__Ex__ : Verify the Divergence Theorem for $\vec{F}(x,y,z) = (z^2+2)\,\vec{k}$ on the top half of the sphere $x^2+y^2+z^2=a^2$ with base the disk $x^2+y^2 \le a^2$ in the xy-plane.