Section 16.8
Thomas Calculus
11th Ed.

Recall: (2D – Space) If $\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$ is a vector field defined on region R, then the divergence of $\vec{F}$ at (x,y) is the scalar function

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} .$$

Def: Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field defined on solid D. The Divergence of $\vec{F}$ at (x,y,z) is the scalar function

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} .$$

Remark: It can be shown that the divergence of $\vec{F}$ at point (x,y,z) is a measure of the tendency of moving fluid to accumulate or disperse at point (x,y,z).

Ex: Find the divergence for

$$\vec{F}(x,y,z) = \underbrace{(e^{x-z}\cos y)}_{M}\vec{i} + \underbrace{(-e^{x-z}\sin y)}_{N}\vec{j} + \underbrace{(x^2 y \tan z)}_{P}\vec{k} :$$

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

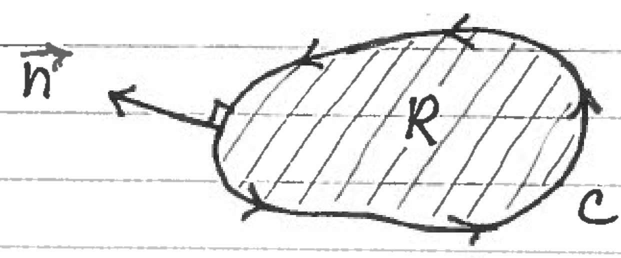
$$= \cancel{e^{x-z}\cos y} - \cancel{e^{x-z}\cos y} + x^2 y \sec^2 z$$

$$= x^2 y \sec^2 z \, .$$

Flux and Divergence in 3D-Space

Recall: (Green's Theorem – Flux Divergence Form – in 2D-Space) assume that $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is defined on region R enclosed by loop C in 2D-Space. Then the flux of $\vec{F}$ across C is

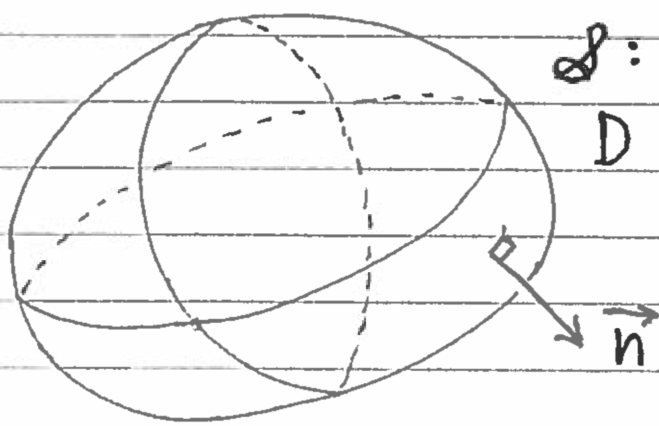
$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \, .$$



The Divergence Theorem will generalize this concept in 3D-Space.

__Divergence Theorem__ : Let $\vec{F}(x,y,z) = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$ be a vector field defined on a solid D, enclosed by surface S with outward-pointing unit normal vector $\vec{n}$. Then the flux of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \underbrace{\vec{\nabla}\cdot\vec{F}}_{\text{div}\,\vec{F}}\,dV .$$

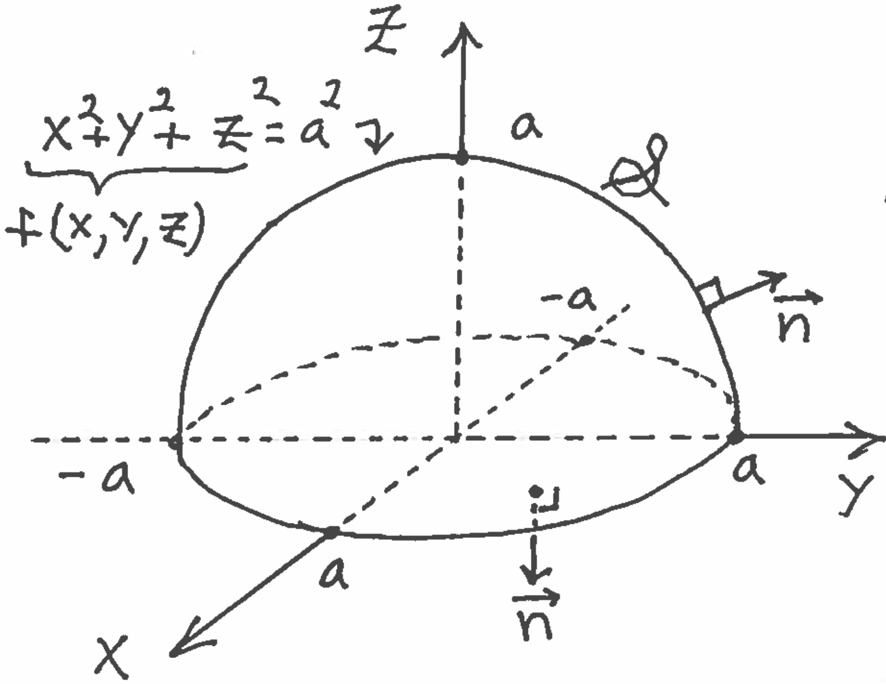


S : surface

D : Solid

__Ex__ : Verify the Divergence Theorem for $\vec{F}(x,y,z) = (z^2+2)\,\vec{k}$ on the top half of the sphere $x^2+y^2+z^2=a^2$ with base the disk $x^2+y^2 \le a^2$ in the xy-plane and $\vec{n}$ is outward pointing normal vector.

$$\operatorname{div} \vec{F} = M_x + N_y + P_z$$
$$= 0 + 0 + 2z = 2k \text{, then:}$$



$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \rho \le a \end{cases}$$

$$\iiint_R \operatorname{div} \vec{F}\, dV = \iiint_R 2z\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a 2(\rho\cos\phi)\cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a 2\rho^3 \cos\phi \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(2\cdot\frac{1}{4}\rho^4 \Big|_{\rho=0}^{\rho=a}\right) \cos\phi \sin\phi \; d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} a^4 \cdot \left(\frac{1}{2} \sin^2\phi \,\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} a^4 \cdot \left(\frac{1}{2} \overset{1}{\cancel{\sin^2\left(\frac{\pi}{2}\right)}} - \frac{1}{2} \overset{0}{\cancel{\sin^2(0)}} \right) d\theta$$

$$= \frac{1}{4} a^4 \cdot \theta \,\Big|_0^{2\pi} = \frac{1}{4} a^4 \cdot (2\pi) = \boxed{\frac{1}{2} a^4 \pi} \;;$$

Surface S is S_1: the top hemi-sphere for $x^2+y^2+z^2=a^2$, and S_2: the disc $x^2+y^2 \le a^2$:

I.) For S_1: $\vec{\nabla} f = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k} \longrightarrow$

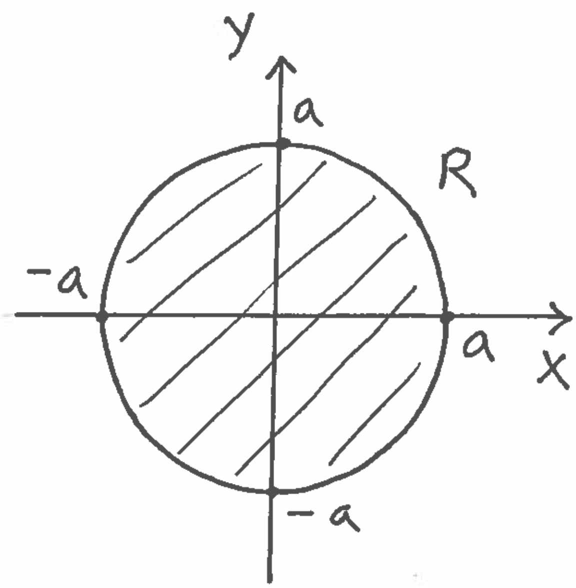
$$|\vec{\nabla} f| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$$

$$= \sqrt{4(x^2+y^2+z^2)}$$

$$= 2\sqrt{a^2} = 2a \,, \quad \text{so}$$

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \left(\frac{x}{a}\right)\vec{i} + \left(\frac{y}{a}\right)\vec{j} + \left(\frac{z}{a}\right)\vec{k} \;;$$

$$R : \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le a \end{cases}$$



$$\iint_{S_1} \vec{F} \cdot \vec{n} \, dS = \iint_{S_1} (z^2 + 2) \frac{z}{a} \, dS$$

$$= \iint_R (z^2 + 2) \frac{z}{a} \cdot \frac{|\vec{\nabla} f|}{|f_z|} \cdot dA$$

$$= \iint_R (z^2 + 2) \frac{\cancel{z}}{\cancel{a}} \cdot \frac{\cancel{2a}}{\cancel{2z}} \, dA$$

$$= \iint_R \left(a^2 - (x^2 + y^2) + 2\right) \, dA$$

$$= \int_0^{2\pi} \int_0^a (a^2 - r^2 + 2) \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^a (a^2 r - r^3 + 2r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(a^2 \cdot \frac{1}{2} r^2 - \frac{1}{4} r^4 + r^2\right)\Big|_{r=0}^{r=a} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} a^4 - \frac{1}{4} a^4 + a^2\right) d\theta$$

$$= \left(\frac{1}{4} a^4 + a^2\right) \theta \Big|_0^{2\pi}$$

$$= \left(\frac{1}{4} a^4 + a^2\right) 2\pi \quad ;$$

II.) For S_2: $\vec{n} = -\vec{k}$ and surface is $\underbrace{z = 0}_{f(x,y,z)}$, so $\vec{\nabla} f = (0)\vec{i} + (0)\vec{j} + (1)\vec{k}$ and $|\vec{\nabla} f| = 1$, so

$$\iint_{S_2} \vec{F} \cdot \vec{n}\, dS = \iint_{S_2} \vec{F} \cdot (-\vec{k})\, dS$$

$$= \iint_{S_2} -(z^2 + 2)\, dS$$

$$= \iint_{S_2} -\left((0)^2 + 2\right) \, dS$$

$$= \iint_R -2 \cdot \frac{|\vec{\nabla} f|}{|f_z|} \, dA$$

$$= \iint_R -2 \cdot \frac{1}{|1|} \, dA$$

$$= -2 \iint_R 1 \, dA$$

$$= -2 \, (\text{Area of } R)$$

$$= -2 \, (\pi a^2) \; ; \quad \textit{finally}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_{S_1} \vec{F} \cdot \vec{n} \, dS + \iint_{S_2} \vec{F} \cdot \vec{n} \, dS$$

$$= \frac{1}{2} a^4 \pi + \cancel{2a^2\pi} - \cancel{2a^2\pi} = \boxed{\frac{1}{2} a^4 \pi} \; .$$