Section 15.1
Thomas Calculus
11th Ed.

Example: Evaluate the following Double Integrals.

1.) $\int_0^1 \int_0^2 (2X-6Y)\,dy\,dx$

$= \int_0^1 \left[2XY-3Y^2\right]\Big|_{Y=0}^{Y=2} dx$

$= \int_0^1 \left((4X-12)-(0-0)\right) dx$

$= \int_0^1 (4X-12)\,dx = (2X^2-12X)\Big|_0^1$

$= (2-12)-(0-0) = -10$

2.) $\int_{-1}^0 \int_0^3 (x^2y+1)\,dy\,dx$

$= \int_{-1}^0 \left[x^2\cdot\frac{1}{2}y^2+y\right]\Big|_{Y=0}^{Y=3} dx$

$= \int_{-1}^0 \left[\left(\frac{9}{2}x^2+3\right)-(0+0)\right] dx$

$= \int_{-1}^0 \left(\frac{9}{2}x^2+3\right) dx = \left(\frac{3}{2}x^3+3x\right)\Big|_{-1}^0$

$= (0+0) - \left(-\frac{3}{2} - 3\right)$

$= \frac{3}{2} + \frac{6}{2} = \frac{9}{2}$

3.) $\int_0^3 \int_{-1}^0 (x^2 y + 1)\, dx\, dy$

$= \int_0^3 \left[\frac{1}{3}x^3 \cdot y + x\right]\Big|_{x=-1}^{x=0} dy$

$= \int_0^3 \left((0+0) - \left(-\frac{1}{3}y - 1\right)\right) dy$

$= \int_0^3 \left(\frac{1}{3}y + 1\right) dy = \left(\frac{1}{3} \cdot \frac{1}{2}y^2 + y\right)\Big|_0^3$

$= \left(\frac{9}{6} + 3\right) - (0+0) = \frac{3}{2} + \frac{6}{2} = \frac{9}{2}$

4.) $\int_1^2 \int_0^2 6 \cdot \sqrt{x+y}\, dy\, dx$

$= \int_1^2 \left[6 \cdot \frac{2}{3}(x+y)^{3/2}\right]\Big|_{y=0}^{y=2} dx$

$= \int_1^2 \left(4(x+2)^{3/2} - 4x^{3/2}\right) dx$

$$= \left(4\cdot\frac{2}{5}(x+2)^{5/2} - 4\cdot\frac{2}{5}x^{5/2}\right)\Big|_1^2$$

$$= \left(\frac{8}{5}(4)^{5/2} - \frac{8}{5}(2)^{5/2}\right) - \left(\frac{8}{5}(3)^{5/2} - \frac{8}{5}\right)$$

$$= \frac{8}{5}(32) - \frac{8}{5}(2)^{5/2} - \frac{8}{5}(3)^{5/2} + \frac{8}{5}$$

$$= \frac{248}{5} - \frac{8}{5}(2)^{5/2} - \frac{8}{5}(3)^{5/2}$$

5.) $\int_0^1 \int_0^{2x} 4xy \, dy \, dx$

$$= \int_0^1 \left(4x\cdot\frac{1}{2}y^2\right)\Big|_{y=0}^{y=2x} dx$$

$$= \int_0^1 \left(2x(4x^2) - 0\right) dx$$

$$= \int_0^1 8x^3 \, dx = 2x^4\Big|_0^1$$

$$= 2(1) - 0 = 2$$

6.) $\int_{-1}^1 \int_0^{y^2} 3x^2 e^{y^7} \, dx \, dy$

$$= \int_{-1}^1 (x^3 e^{y^7})\Big|_{x=0}^{x=y^2} dy$$

$$= \int_{-1}^{1} \left(y^6 e^{y^7} - 0\right) dy$$

$$= \frac{1}{7} e^{y^7} \Big|_{-1}^{1} = \frac{1}{7} e - \frac{1}{7} e^{-1}$$

7.) $$\int_0^2 \int_0^{\sqrt{y}} \frac{2x}{(1+x^2)(1+y)}\, dx\, dy$$

$$= \int_0^2 \left(\frac{1}{1+y} \ln(1+x^2)\right)\Big|_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^2 \left(\frac{1}{1+y} \ln(1+y) - \frac{1}{1+y} \overset{0}{\cancel{\ln 1}}\right) dy$$

$$= \int_0^2 \frac{1}{1+y} \ln(1+y)\, dy$$

$$= \frac{1}{2}\left(\ln(1+y)\right)^2 \Big|_0^2$$

$$= \frac{1}{2}(\ln 3)^2 - \frac{1}{2}(\overset{0}{\cancel{\ln 1}})$$

$$= \frac{1}{2}(\ln 3)^2$$

8.) $\int_0^{\ln 2} \int_0^{\ln x} 4e^{2x+y}\, dy\, dx$

$$= \int_0^{\ln 2} \left(4e^{2x+y}\right)\Big|_{y=0}^{y=\ln x} dx$$

$$= \int_0^{\ln 2} \left(4e^{2x+\ln x} - 4e^{2x}\right) dx$$

$$= \int_0^{\ln 2} \left(4e^{2x} \cancel{e^{\ln x}}^{\,x} - 4e^{2x}\right) dx$$

$$= \int_0^{\ln 2} 4xe^{2x}\, dx - 4\int_0^{\ln 2} e^{2x}\, dx$$

$\Big($ Let $u = 4x$, $dv = e^{2x}\, dx$

$\rightarrow du = 4\, dx$, $v = \frac{1}{2}e^{2x} \Big)$

$$= \left(2xe^{2x}\Big|_0^{\ln 2} - 2\int_0^{\ln 2} e^{2x}\, dx\right)$$

$$- 4 \cdot \frac{1}{2} e^{2x}\Big|_0^{\ln 2}$$

$$= \left(2\ln 2\, e^{2\ln 2} - 0\right) - 2 \cdot \frac{1}{2} e^{2x}\Big|_0^{\ln 2}$$

$$- \left(2e^{2\ln 2} - 2e^{0}\right)$$

$$= 2\ln 2 \cdot e^{\ln 2^2} - \left(e^{2\ln 2} - e^{0}\right)$$

$$- 2e^{\ln 2^2} + 2(1)$$

$$= 2\ln 2 \cdot (4) - 4 + 1 - 2(4) + 2$$

$$= 8\ln 2 - 9$$

9.) $$\int_0^{\pi}\int_0^{\frac{\pi}{3}} \sin(x-y)\, dy\, dx$$

$$= \int_0^{\pi} \left(\cos(x-y)\right)\Big|_{y=0}^{y=\frac{\pi}{3}} dx$$

$$= \int_0^{\pi} \left(\cos\left(x-\frac{\pi}{3}\right) - \cos x\right) dx$$

$$= \left(\sin\left(x-\frac{\pi}{3}\right) - \sin x\right)\Big|_0^{\pi}$$

$$= \left(\sin\left(\pi-\frac{\pi}{3}\right) - \overset{0}{\sin \pi}\right)$$

$$- \left(\sin\left(-\frac{\pi}{3}\right) - \overset{0}{\sin 0}\right)$$

$$= \sin\left(\frac{2}{3}\pi\right) - \left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$$