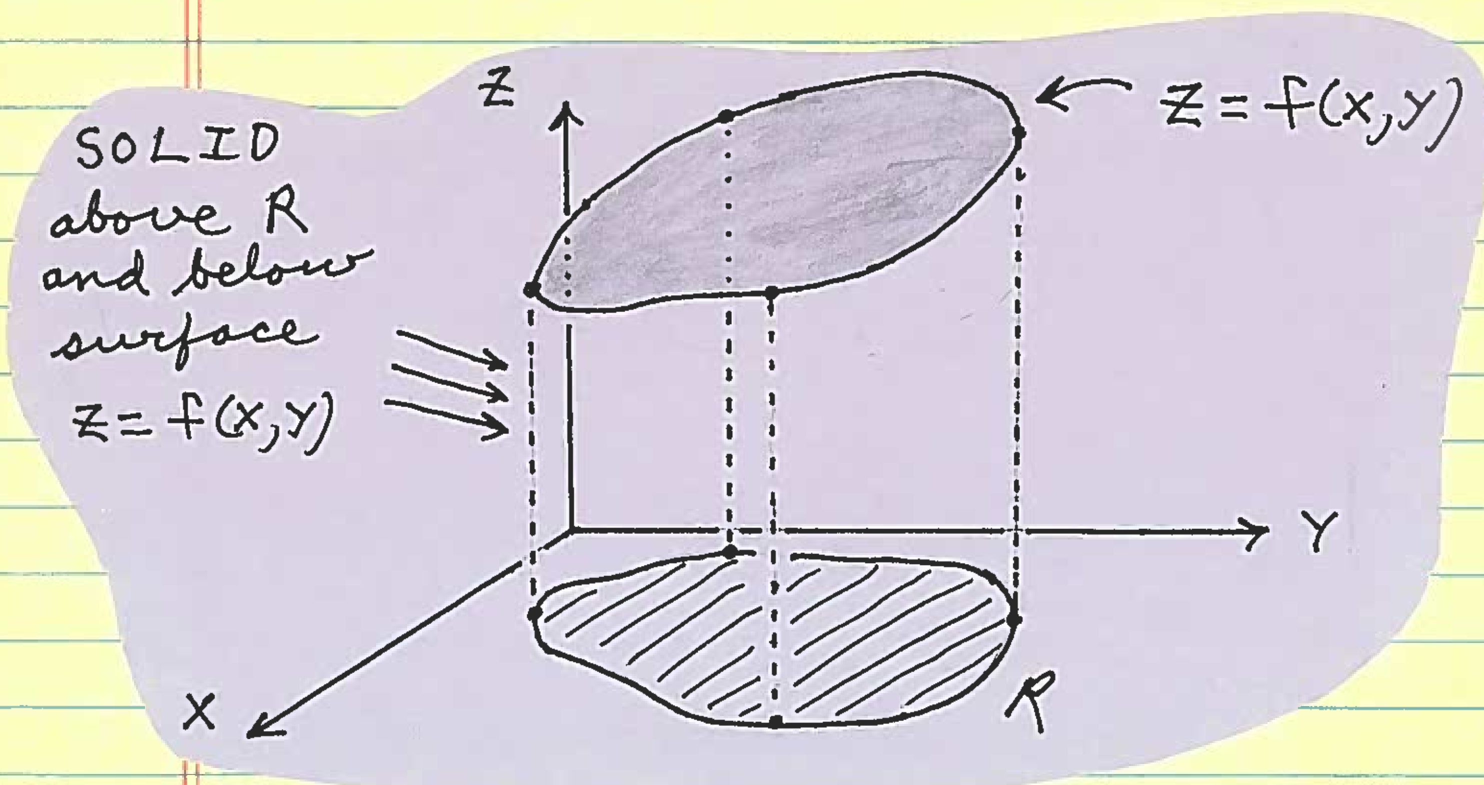
Section 15.1
Thomas Calculus
11th Ed.

Volume

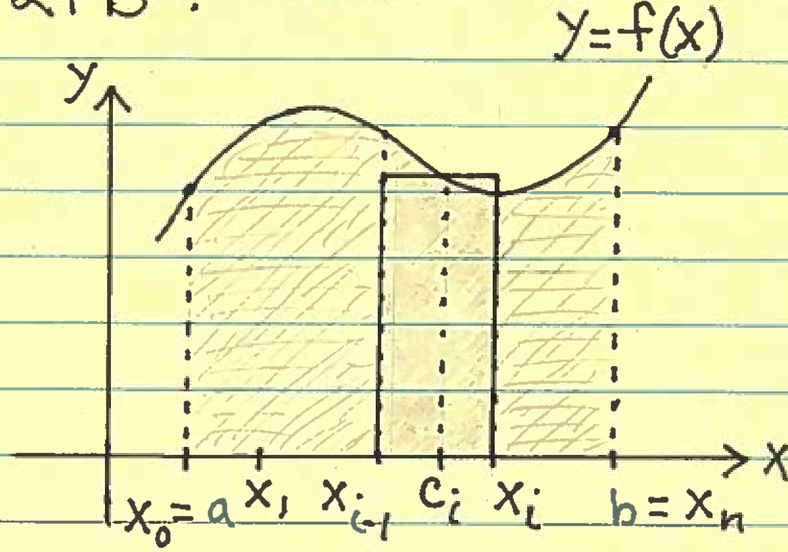
Consider the surface $z = f(x, y)$ in 3D-Space above the region R in the xy-plane (See sketch). How do we compute the Volume of the resulting solid?



Before we answer this question, let's recall the Formal Definition of the Definite Integral $\int_a^b f(x)\,dx$ from Math 21B :

1.) <u>Partition</u> interval $[a,b]$ into n parts :

$a=x_0, x_1, x_2, \ldots, x_n=b$



2.) Choose a random <u>sampling point</u> c_i in $[x_{i-1}, x_i]$ for $i=1,2,3,\ldots,n$ and $\Delta x_i = x_i - x_{i-1}$

3.) Define the <u>mesh</u> of the partition ~~to~~ be

$$\text{mesh} = \max_{1 \le i \le n} (x_i - x_{i-1})$$

(i.e., the *mesh* is the largest of the distances between consecutive partition points)

4.) The Definite Integral is:

$$\int_a^b f(x)\,dx = \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

Remarks: i.) $\sum_{i=1}^{n} f(c_i)\,\Delta x_i$ is the SUM of the areas of n rectangles, so that $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x_i$

ii.) " $\lim_{\text{mesh}\to 0}$ " ensures that the # of rectangles $n \to \infty$ and the estimate becomes exact, i.e.,

$$\int_a^b f(x)\,dx = \lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(c_i)\cdot \Delta x_i$$

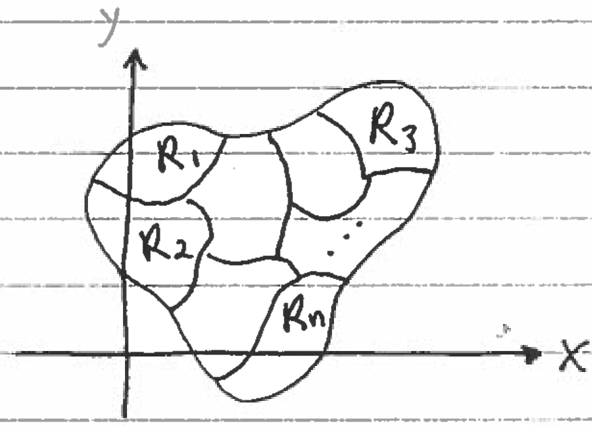
The Definite Integral of $z = f(x,y)$ Over Region R in the Plane

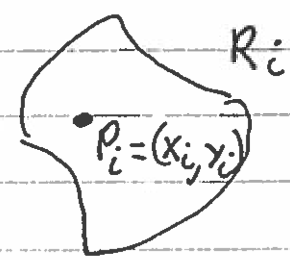
SEE the following 4 page handout.

Math 21D
Kouba

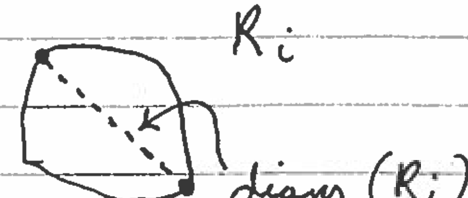
The Definite Integral Over Regions in the Plane

Let $z = f(x, y)$ be a function of two variables defined on a region R in the xy-plane. Partition R into n parts $R_1, R_2, \ldots, R_n$ of areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, resp. Let $P_i = (x_i, y_i)$ be an arbitrary point in R_i for $i = 1, 2, 3, \ldots, n$.





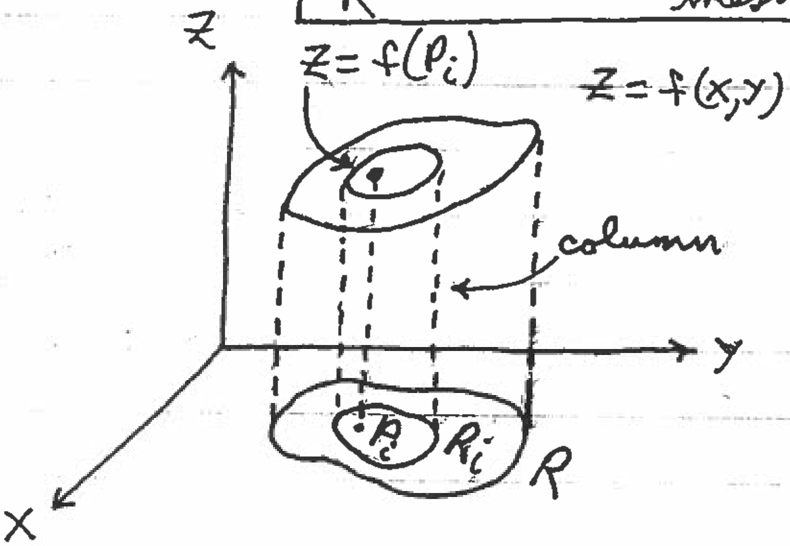
Define the diameter of each part R_i, $\text{diam}(R_i)$, to be the largest possible distance between any two points of R_i for $i = 1, 2, 3, \ldots, n$.
Define the mesh of the partition to be the largest of the n diameters, i.e.,



$$\text{mesh} = \max_{1 \le i \le n} (\text{diam}(R_i)).$$

Define the <u>definite integral</u> of $z = f(x,y)$ over the region R to be

$$\boxed{\int_R f(P)\, dA = \lim_{\text{mesh} \to 0} \sum_{i=1}^{n} f(P_i) \cdot \Delta A_i}.$$



The area of R_i is ΔA_i so $f(P_i) \cdot \Delta A_i$ is an estimate for the volume of the column.

<u>Remarks</u>: 1.) $\int_R 1\, dA$ = area of R.

2.) If $f(P)$ measures depth of solid at point P, then

$$\int_R f(P)\, dA = \text{Volume of Solid}.$$

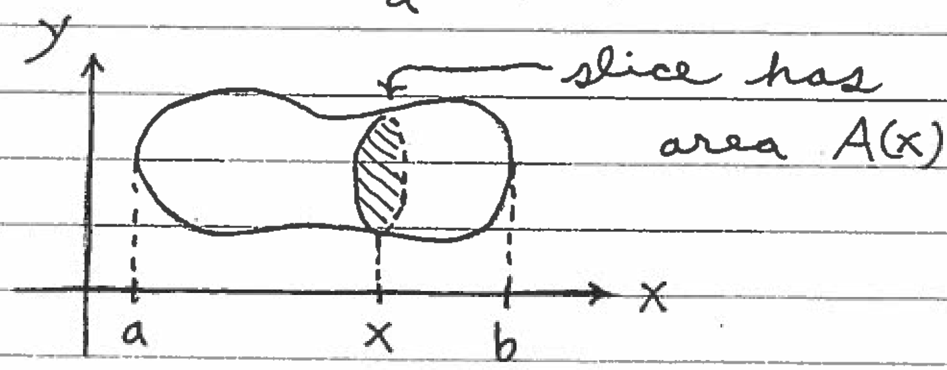
3.) If $f(P)$ measures density ($\frac{\text{mass}}{\text{area}}$ units) at point P, then

$$\int_R f(P)\, dA = \text{Mass of Flat Region } R.$$

Evaluating $\int_R f(P)\,dA$ Using Rectangular Coordinates

Recall: If a solid has a known cross-sectional area $A(x)$ if slice is made perpendicular to the x-axis at x, then the volume of the solid is

$$\text{Volume} = \int_a^b A(x)\,dx.$$

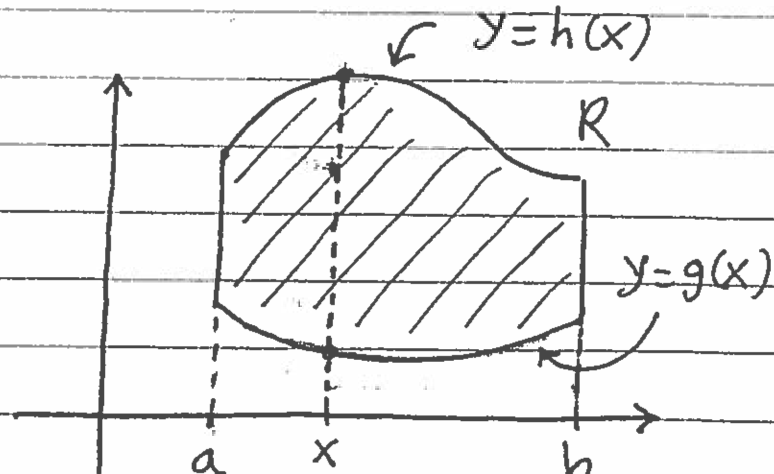


Let region R in the plane be given by

$$a \le x \le b$$

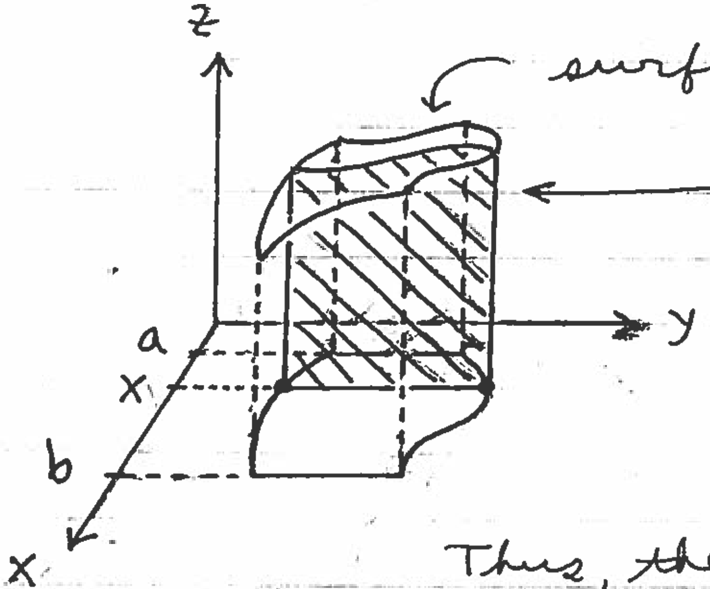
and

$$g(x) \le y \le h(x)$$



Let $z = f(x,y)$ be a function of two variables. We seek a method to evaluate

$$\int_R f(P)\,dA.$$

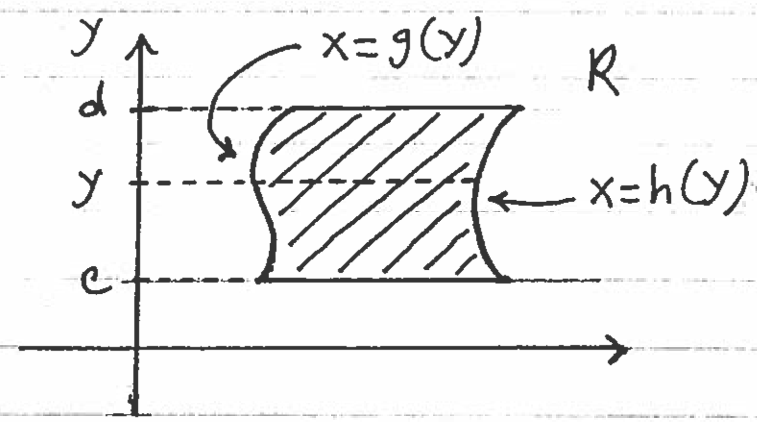


This slice has area

$$A(x) = \int_{g(x)}^{h(x)} f(x,y)\,dy.$$

Thus, the volume of the solid is

$$\int_R f(P)\,dA = \int_a^b A(x)\,dx$$
$$= \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx.$$



Let region R in the plane be given by

$$c \le y \le d$$

and

$$g(y) \le x \le h(y).$$

It follows analogously that

$$\int_R f(P)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$