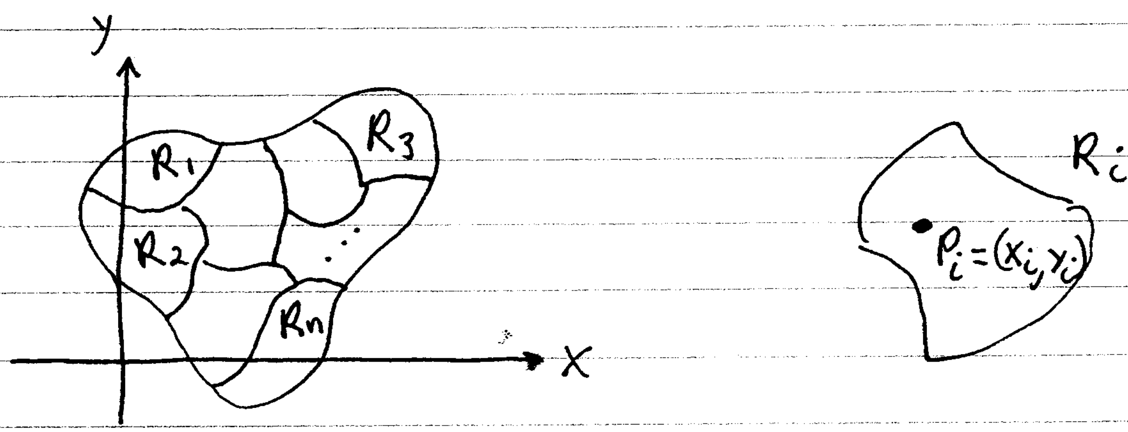
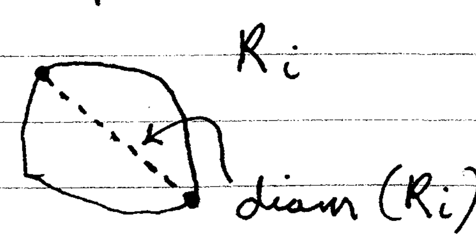
Math 21D
Kouba

The Definite Integral Over Regions in the Plane

Let $z = f(x,y)$ be a function of two variables defined on a region R in the xy-plane. Partition R into n parts $R_1, R_2, \ldots, R_n$ of areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, resp. Let $P_i = (x_i, y_i)$ be an arbitrary point in R_i for $i = 1, 2, 3, \ldots, n$.



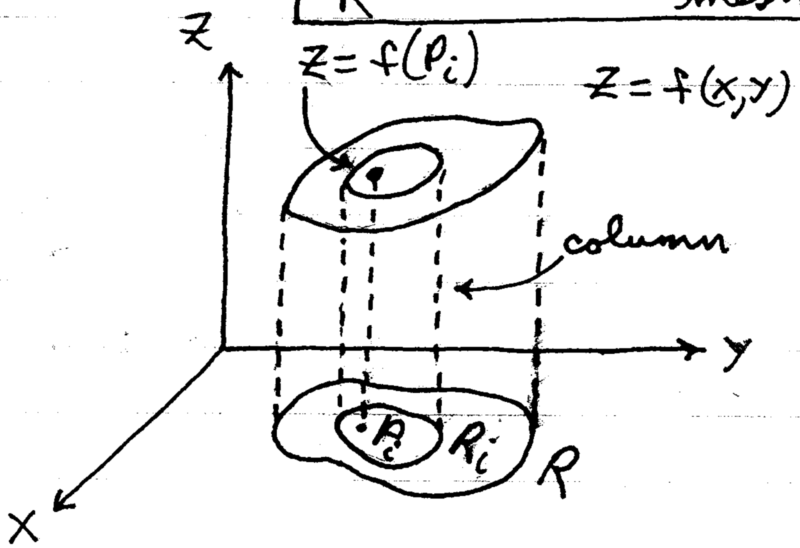
Define the diameter of each part R_i, $\text{diam}(R_i)$, to be the largest possible distance between any two points of R_i for $i = 1, 2, 3, \ldots, n$.
Define the mesh of the partition to be the largest of the n diameters, i.e.,



$$\text{mesh} = \max_{1 \le i \le n} (\text{diam}(R_i)).$$

Define the <u>definite</u> <u>integral</u> of $z = f(x,y)$ over the region R to be

$$\boxed{\int_R f(P)\, dA = \lim_{\text{mesh} \to 0} \sum_{i=1}^{n} f(P_i) \cdot \Delta A_i}.$$



The area of R_i is ΔA_i so $f(P_i) \cdot \Delta A_i$ is an estimate for the volume of the column.

<u>Remarks</u>: 1.) $\int_R 1\, dA$ = area of R.

2.) If $f(P)$ measures depth of solid at point P, then

$$\int_R f(P)\, dA = \text{Volume of Solid}.$$

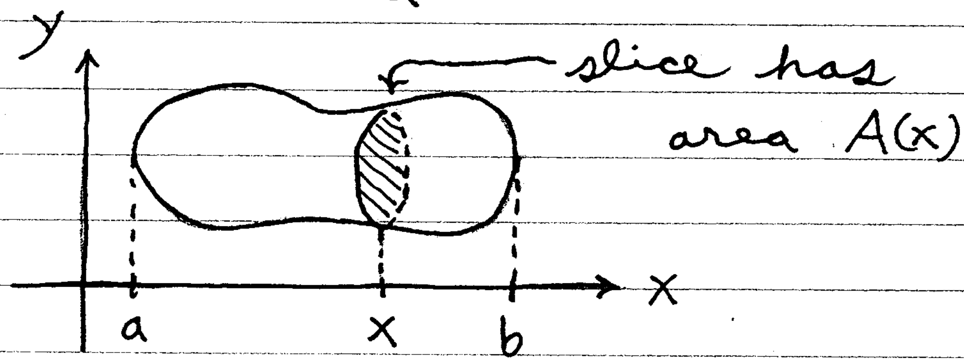
3.) If $f(P)$ measures density ($\frac{\text{mass}}{\text{area}}$ units) at point P, then

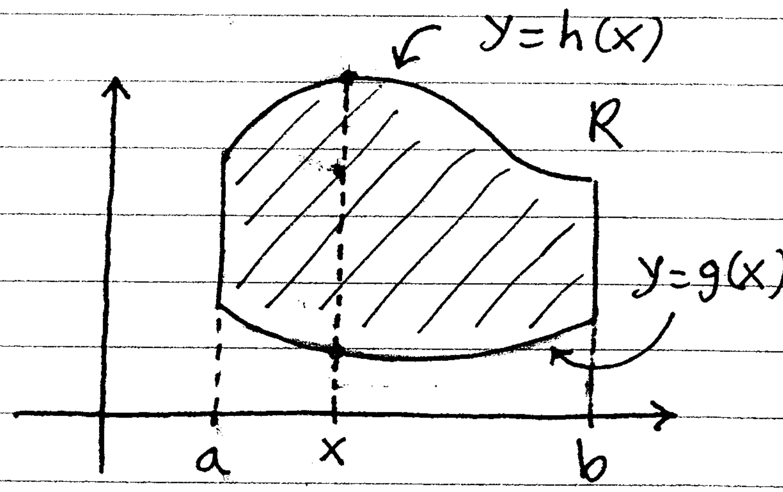
$$\int_R f(P)\, dA = \text{Mass of Flat Region } R.$$

Evaluating $\int_R f(P)\, dA$ Using Rectangular Coordinates

Recall: If a solid has a known cross-sectional area $A(x)$ if slice is made perpendicular to the x-axis at x, then the volume of the solid is

$$\text{Volume} = \int_a^b A(x)\, dx.$$





Let region R in the plane be given by

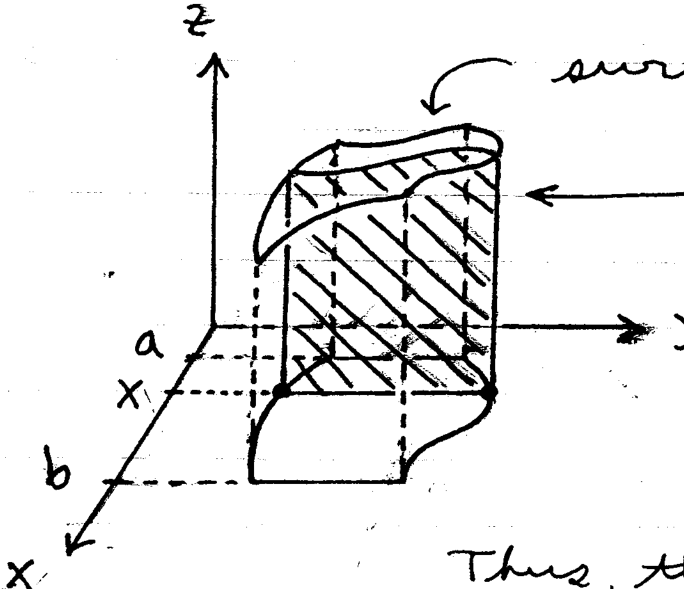
$$a \le x \le b$$

and

$$g(x) \le y \le h(x).$$

Let $z = f(x, y)$ be a function of two variables. We seek a method to evaluate

$$\int_R f(P)\, dA.$$



This slice has area

$$A(x) = \int_{g(x)}^{h(x)} f(x,y)\,dy .$$

Thus, the volume of the solid is

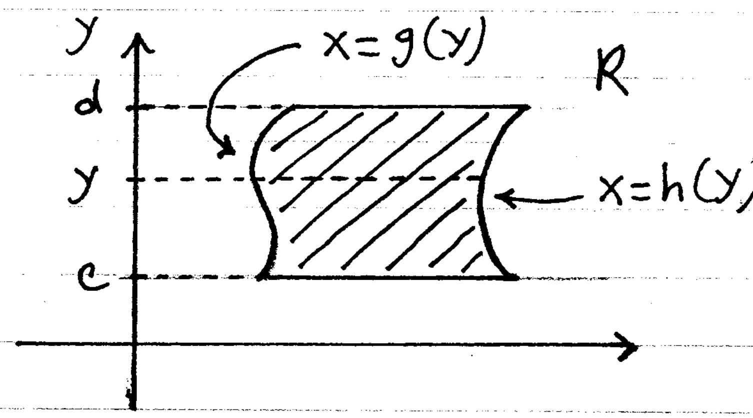
$$\int_R f(P)\,dA = \int_a^b A(x)\,dx$$
$$= \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx .$$

Let region R in the plane be given by

$$c \le y \le d$$

and

$$g(y) \le x \le h(y) .$$



It follows analogously that

$$\int_R f(P)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$