The Definite Integral Over Regions in the Plane

Let \( z = f(x, y) \) be a function of two variables defined on a region \( R \) in the \( xy \)-plane. Partition \( R \) into \( n \) parts \( R_1, R_2, \ldots, R_n \) of areas \( AA_1, AA_2, \ldots, AA_n \), resp. Let \( P_i = (x_i, y_i) \) be an arbitrary point in \( R_i \) for \( i = 1, 2, 3, \ldots, n \).

Define the diameter of each part \( R_i \), \( \text{diam} (R_i) \), to be the largest possible distance between any two points of \( R_i \) for \( i = 1, 2, 3, \ldots, n \). Define the mesh of the partition to be the largest of the \( n \) diameters, i.e.,
Define the definite integral of \( z = f(x,y) \) over the region \( R \) to be

\[
\int_R f(p) \, dA = \lim_{\text{mesh} \to 0} \sum_{i=1}^{n} f(p_i) \cdot \Delta A_i.
\]

The area of \( R_i \) is \( \Delta A_i \) so \( f(p_i) \cdot \Delta A_i \) is an estimate for the volume of the column.

Remarks:
1.) \( \int_R 1 \, dA = \text{area of } R \).

2.) If \( f(p) \) measures depth of solid at point \( p \), then
\[
\int_R f(p) \, dA = \text{Volume of Solid}.
\]

3.) If \( f(p) \) measures density \( \left( \frac{\text{mass}}{\text{area unit}} \right) \) at point \( p \), then
\[
\int_R f(p) \, dA = \text{Mass of Flat Region } R.
\]
Evaluating $\int_R f(p) \, dA$ Using Rectangular Coordinates

Recall: If a solid has a known cross-sectional area $A(x)$ if slice is made perpendicular to the x-axis at $x$, then the volume of the solid is

$$\text{Volume} = \int_a^b A(x) \, dx.$$
Let region $R$ in the plane be given by $c \leq y \leq d$ and $g(y) \leq x \leq h(y)$.

It follows analogously that

$$
\int_{R} f(x,y) \, dx \, dy = \int_{c}^{d} \int_{g(y)}^{h(y)} f(x,y) \, dx \, dy
$$